Mechanics

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1 Mathmatical Preparation

The most important ideas in mechanics are forces and moments, which are mathematically treated as **vectors**. The most fundamental law in mechanics is Newton's law of motion, which is described in the form of **differential equations**. Therefore, when learning the fundamentals of mechanics, we need the mathematical basis for vectors and differential equations. At first, we thus learn about the basis of vectors and differential equations.

1.1 Sum of vectors

The sum of two vector $\mathbf{A} + \mathbf{B}$ is defined by using each elements as [1]

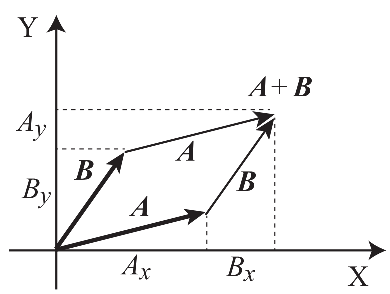


Figure 1.1: Composition and decomposition of vectors (two dimensional case).

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = (A_x, A_y, A_z)$$
$$\boldsymbol{B} = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k} = (B_x, B_y, B_z)$$
$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\boldsymbol{i} + (A_y + B_y)\boldsymbol{j} + (A_z + B_z)\boldsymbol{k} = (A_x + B_x, A_y + B_y, A_z + B_z) \tag{1.1}$$

The two vectors can be composed and decomposed using a parallelogram as in Fig. 1. Where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are **unit vectors** along X-axis, Y-axis, Z-axis.

[1] The vectors in this Mechanics textbook are three dimensional vectors, which has three elements for each vector.

1.2 Inner product of vectors

The **inner product** (or **scalar product**) for vector $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} \triangleq |\boldsymbol{A}||\boldsymbol{B}| \cos\theta \tag{1.2}$$

where $|\boldsymbol{A}|$ is the length of vector $\boldsymbol{A}$. Note that the inner product has a **scalar** value.

The inner product can be also calculated by

$$\boldsymbol{A} \cdot \boldsymbol{B} = A_x B_x + A_y B_y + A_z B_z \tag{1.3}$$

(Problem 1.1)
Prove Equation (1.1) by denoting the vector can be expressed by

$$\boldsymbol{A} = A_x \boldsymbol{i} + A_y \boldsymbol{j} + A_z \boldsymbol{k} \tag{1.4}$$

($\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is called **unit orthogonal vectors**.)

(Solutuion for Problem 1.1)

$$\boldsymbol{A} \cdot \boldsymbol{B} = (A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}) \cdot (B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}) = A_x B_x + A_y B_y + A_z B_z \quad (\boldsymbol{i} \cdot \boldsymbol{i} = 0, \boldsymbol{i} \cdot \boldsymbol{j} = 1, \cdots)$$

(Problem 1.2)
Prove

$$|\boldsymbol{A}| = A_x^2 + A_y^2 + A_z^2 \quad = \sqrt{\boldsymbol{A} \cdot \boldsymbol{A}} \;\; (= A)$$

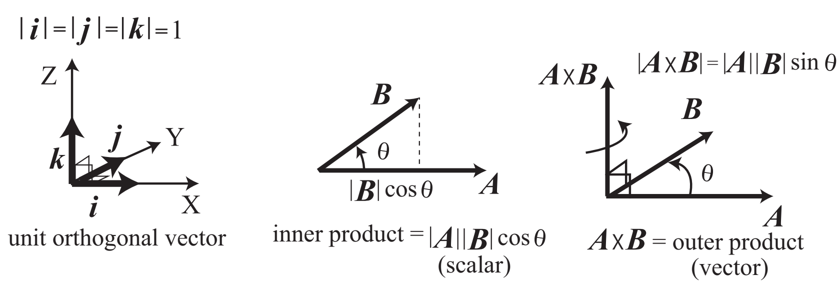


Figure 1.2: Definition of inner and outer product

1.3 Outer product of vectors

The **outer product** (or **vector product**) for vector $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined by the magnitude and the direction (because the outer product is also a vector) as

$$\text{magnitude of } \boldsymbol{A} \times \boldsymbol{B} : \quad |\boldsymbol{A} \times \boldsymbol{B}| \triangleq |\boldsymbol{A}||\boldsymbol{B}| \sin\theta \tag{1.5}$$

$$\text{direction of } \boldsymbol{A} \times \boldsymbol{B} : \quad \text{the screw orthogonal direction from } \boldsymbol{A} \text{ to } \boldsymbol{B} \;\; \text{(see Figure 1.2)} \tag{1.6}$$

(Problem 1.3)
Prove

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j}, \quad \boldsymbol{j} \times \boldsymbol{i} = -\boldsymbol{k}$$

(Problem 1.4)
Prove

$$\boldsymbol{A} \times \boldsymbol{B} = (A_y B_z - A_z B_y)\boldsymbol{i} - (A_x B_z - A_z B_x)\boldsymbol{j} + (A_x B_y - A_y B_x)\boldsymbol{k}$$

1.4 Differential and integral of vectors

The time differential of a vector $\boldsymbol{r}(t)$ (velocity $\boldsymbol{v}(t)$) can be calculated as the differential of each element as

$$\frac{d\boldsymbol{r}(t)}{dt} = \dot{\boldsymbol{r}} = \frac{dr_x(t)}{dt}\boldsymbol{i} + \frac{dr_y(t)}{dt}\boldsymbol{j} + \frac{dr_z(t)}{dt}\boldsymbol{k} = \boldsymbol{v} \tag{1.7}$$

(Problem 1.5)
Prove Equation 1.7 using the following definition of differential

$$\frac{d\boldsymbol{r}(t)}{dt} \triangleq \lim_{h\to 0} \frac{\boldsymbol{r}(t+h) - \boldsymbol{r}(t)}{h} \tag{1.8}$$

Similarly the integral of a vector $\boldsymbol{r}$ is calculated as the integral of each element by

$$\int \boldsymbol{r}(t)dt = \int r_x(t)dt\ \boldsymbol{i} + \int r_y(t)dt\ \boldsymbol{j} + \int r_z(t)dt\ \boldsymbol{k} \tag{1.9}$$

(Problem 1.6)
Prove for the following derivative on the inner product $\boldsymbol{A}\cdot\boldsymbol{B}$

$$\frac{d(\boldsymbol{A}\cdot\boldsymbol{B})}{dt} = \frac{d\boldsymbol{A}}{dt}\cdot\boldsymbol{B} + \boldsymbol{A}\cdot\frac{d\boldsymbol{B}}{dt}$$

(Problem 1.7)
Prove for the following derivative on the outer product $\boldsymbol{A}\times\boldsymbol{B}$

$$\frac{d(\boldsymbol{A}\times\boldsymbol{B})}{dt} = \frac{d\boldsymbol{A}}{dt}\times\boldsymbol{B} + \boldsymbol{A}\times\frac{d\boldsymbol{B}}{dt}$$

1.5 Solution of first order linear differential equation

In mechanics, we need to solve some **differential equations**. The **first order time-invariant constant coefficient** differential equation is witten by

$$\frac{dx}{dt} + ax = u(t) \tag{1.10}$$

where a is a constant, t is a **independent variable** and x is **dependent variable** ($x = x(t)$). Note that the differential equation is usually described by

$$\frac{dy}{dx} + ay = u(x)$$

in mathematical manner. At first we solve the differential equation (D.E.) for the case of $u(t) = 0$, which is called a **homogeneous differential equation** as

$$\frac{dx}{dt} + ax = 0 \tag{1.11}$$

Solving the differential equation is to find a function $x = x(t)$ which is satisfied with the original differential equation.The most basic method to solve the differential equation is **"variable separation"**, which is

$$\frac{dx}{dt} = -ax \ \Rightarrow\ \frac{dx}{x} = -a \ \Rightarrow\ \int\frac{dx}{x} = -a\int dt \tag{1.12}$$

By integrating for the both side,

$$\ln|x| + C = -at + C' \ \Rightarrow\ |x| = e^{-at+C} \ \Rightarrow\ x = \pm C\times e^{at} = Ce^{-at} \tag{1.13}$$

This solution is called **"basic solution"** for the homogeneous differential equation. In general the first order differential equation contains one **"unknown constant"** C. We next calculate for the case of $u(t) \neq 0$, which is called **non-homogeneous differential equation**. The general structure of the solution for the non-homogeneous differential equation is

$$x(t) = (\text{constant}) \times (\text{homogeneous basic solution}) + (\text{special solution}) \tag{1.14}$$

where **special solution** is a solution satisfying with the "original non-homogeneous differential equation". Note that the homogeneous basic solution is not a solution for the non-homogeneous differential equation. There is no definite way to get the special solution for general **non-homogeneous term** $u(t)$. It is, however, easy to get the special solution for many non-homogeneous term $u(t)$. For example $u(t) =$ "constant" say A then special solution is also a constant B. (The values are different.) When $u(t) = at + b$ then the special solution is $ct + d$. When $u(t) = A \sin at$, then the special solution is $B \sin at + C \cos at$. When $u(t) = Ae^{at}$ then the special solution is Be^{at}.

(Problem 1.8)
Solve the following differential equations.

$$\frac{dx}{dt} + 3x = 2e^{2t}$$

$$\frac{dx}{dt} + x = \cos 2t$$

In mathematics, the independent variable is normally x and dependent variable for that is y ($y = y(x)$). In mechanics (or in physics), independent variable is usually time t, and dependent variable is x (this variable may be various with physical meaning). Furthermore, the derivative on t for x is often written by $\dot{x}$.

1.6 Solution of second order linear differential equation (real solution)

We now solve the following differential equation (homogeneous one).

$$\ddot{x} + a\dot{x} + bx = 0 \tag{1.15}$$

where $\ddot{x} = \dfrac{d^2x}{dt^2}$, $\dot{x} = \dfrac{dx}{dt}$, a, b are constants. As an example for this type of differential equation

$$\ddot{x} + 3\dot{x} + 2x = 0 \tag{1.16}$$

As we have already known the linear constant coefficient time-invariant differential equation has the solution of the form $e^{\lambda t}$. So we set the solution for Equation 1.16 can be described as

$$x(t) = Ce^{\lambda t} \tag{1.17}$$

Substituting Equation 1.17 into Equation 1.16 becomes

$$(\lambda^2 + 3\lambda + 2)Ce^{\lambda t} = 0 \tag{1.18}$$

Generally ($Ce^{\lambda t} \neq 0$) the following equation should be satisfied

$$\lambda^2 + 3\lambda + 2 = (\lambda + 2)(\lambda + 1) = 0 \tag{1.19}$$

which means $\lambda_1 = -2$ or $\lambda_2 = -1$ and the polynomial equation of λ is called **characteristic equation**. Thus we can express the general solution for Equation 1.16 is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} = C_1 e^{-2t} + C_2 e^{-t} \tag{1.20}$$

Actually we can easily confirm the solution satisfies the original differential equation.

(Problem 1.9)
When the second order differential equation has two basic solutions $x_1(t)$ and $x_2(t)$, prove that the solution $\hat{x}(t) = C_1 x_1(t) + C_2 x_2(t)$ is also the solution.

We next consider the following non homogeneous second order differential equation,

$$\ddot{x} + a\dot{x} + bx = u(x) \tag{1.21}$$

As we can imagine, the general solution for the second order non-homogeneous differentials equation has the form as

$$y = \text{(homogeneous solution)} + \text{(special solution)} = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \text{(special solution)} \tag{1.22}$$

As an example

$$\ddot{x} - \dot{x} - 6x = e^{-t}$$

The general solution is

$$x(t) = C_1 e^{-2t} + C_2 e^{3t} - \frac{1}{6} e^{-t}$$

The idea of how to get the special solution is almost same as the case of first order differential equation.

1.7 Solution of second order linear differential equation (simple vibration)

We now consdier the following differential equation,

$$\ddot{x} + \omega^2 x = 0 \tag{1.23}$$

Applying the idea of characteristic equation method,

$$\lambda^2 + \omega^2 = 0 \quad \Rightarrow \lambda = \pm \omega i \tag{1.24}$$

where i is a imaginary unit. The solution, thus, can be written

$$x(t) = C_1 e^{\omega i t} + C_2 e^{-\omega i t} \tag{1.25}$$

Using the Euler's thorem $e^{i\theta} = \cos\theta + i \sin\theta$ (see Appendix A), we have

$$x(t) = C_1(\cos\omega t + i \sin\omega t) + C_2(\cos\omega t - i \sin\omega t) = A \sin\omega t + B \cos\omega t = D \cos(\omega t - \alpha) \tag{1.26}$$

where $D = \sqrt{A^2 + B^2}$ and $\tan\alpha = A/B$. We should note it has two unknown parameters because the original differential equation is the second order one.

1.8 Differential for multi variable function (partial derivative and total derivative)

When we have a multi variable function such as

$$f(\boldsymbol{r}) = f(x, y, z) \tag{1.27}$$

which is, for example, temperature for a specific point (x, y, z) in three dimensional space. Then we have two types of derivative. One is **partial derivative** such as

$$\frac{\partial f}{\partial x} \triangleq \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h} \tag{1.28}$$

Similarly $\dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$ are also defined.

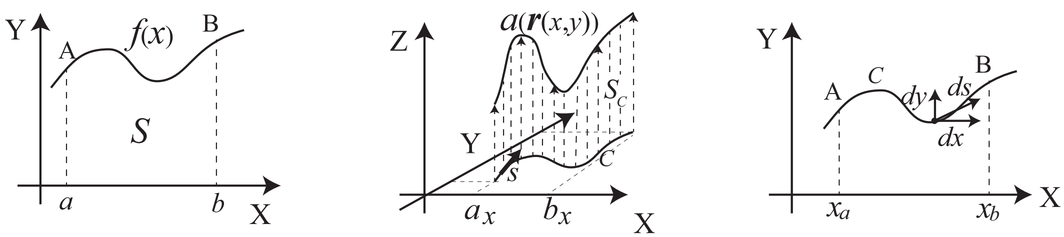
Another is **total derivative**. For example, when x, y, z are time function as $x(t), y(t), z(t)$, then the derivative of y on time t is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \tag{1.29}$$

This is called total derivetive on time t. Or df (total derivative) is discribed by

$$df \triangleq \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \tag{1.30}$$

1.9 Various types of definite integrals



(a) normal definite integral (b) line integral along curve C (c) length of curve C

Figure 1.3: Normal definite integral and line integral

Normal definite integral

$$S = \int_a^b y(x)dx \tag{1.31}$$

S is the area in Figure 1.3(a).

Line integral of scalar function $a(\boldsymbol{r})$

$$S_C = \int_C a(\boldsymbol{r}(s))ds = \int_C a(\boldsymbol{r}(x,y))dxdy = \int_C a(x(s), y(s))ds \tag{1.32}$$

where s is the length on the curve C. This integral means the area of the "curtain" in Figure 1.3(b). When the scalar function $a(\boldsymbol{r}(s))$ is 1, then the integral means the **length** of the curve C itself, which is calculated by

$$l = \int_C ds = \int_C \sqrt{dx^2 + dy^2} = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \tag{1.33}$$

This is the case of two-dimension. For the case of three-dimension,

$$S_C = \int_C a(\boldsymbol{r}(s))ds = \int_C a(x(s), y(s), z(s))ds \tag{1.34}$$

When $a(\mathbf{r}) = 1$,

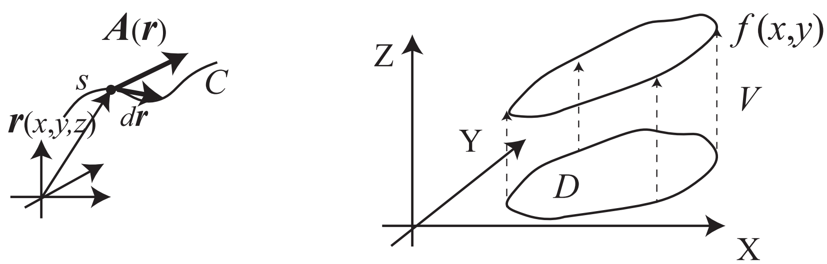
$$l = \int_C ds = \int_C \sqrt{dx^2 + dy^2 + dz^2} = \int_{x_a}^{x_b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2} dx \tag{1.35}$$

Line integral of vector function $\boldsymbol{A}(\boldsymbol{r})$

The line integral of a vector function $\boldsymbol{A}(\boldsymbol{r})$ can be defined using inner product and outer product as

$$\int_C \boldsymbol{A}(\boldsymbol{r}) \cdot d\boldsymbol{r} = \int_C A_x dx + \int_C A_y dy + \int_C A_z dz \tag{1.36}$$

$$\int_C \boldsymbol{A}(\boldsymbol{r}) \times d\boldsymbol{r} = \left\{ \int_C A_y dz - \int_C A_z dy \right\} \boldsymbol{i} - \left\{ \int_C A_x dz - \int_C A_z dx \right\} \boldsymbol{j} + \left\{ \int_C A_x dy - \int_C A_y dx \right\} \boldsymbol{k} \tag{1.37}$$



(a) line integral for vector function $\boldsymbol{A}(\boldsymbol{r})$ along C (b) area integral for scalar function $f(x,y)$ on region D

Figure 1.4: Line integral for vector function $\boldsymbol{A}$ and area integral for scalar function $f(x, y)$

where $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ and $d\boldsymbol{s} = dx\boldsymbol{i} + dy\boldsymbol{j} + dz\boldsymbol{k}$.

Example 1.1 (line integral)

Calculate the line integral of curve C integral for the scalar function $a(x, y, z) = x + 2yz$. where $C : \boldsymbol{s} = s\boldsymbol{i} + s\boldsymbol{j} + s\boldsymbol{k} \quad (0 \le s \le 1)$.

Answer of Example 1.1: Because $x = y = z = s$,

$$a = s + 2s^2 \quad \Rightarrow \quad \int_C a(s)ds = \int_0^1 (s + 2s^2)ds = \frac{7}{6}$$

Area integral for a scalar function $f(x, y, z)$

The area integral for a scalar function $f(x, y)$ for the region D (two dimension) is

$$\int_D f(x, y)ds = \int_D f(x, y)dxdy \tag{1.38}$$

This means the volume V in Figure 1.4(b). When the case of $f(x, y) = 1$, this means just the area of region D. For three dimension case,

$$\int_D f(x, y, z)ds = \int_D f(x, y, z)dxdydz \tag{1.39}$$

where the area D is a three dimensional surface. The area integral for vector functions are also defined as for the case of line integral.

Volume integral for a scalar function $g(x, y, z)$

The volume integral for a scalar function $g(x, y, z)$ for the region V (a three dimensional volume) is

$$\int_V g(x, y, z)dv = \int_V g(x, y, z)dxdydz \tag{1.40}$$

When the case of $g(x, y, z) = 1$, this means just the volume of region V. The volume integral for a vector function is also defined as for the case of line integral. The integral calculation for multiple variable such as $\int f(x,y)dxdy$ and $\int g(x,y,z)dxdydz$ are called **multiple integration**.

Example 1.2 (area integral, multiple integral)
Calculate the following area integral for a scalar function $f(x,y) = x^2$ on region $D = D_1$ or D_2.

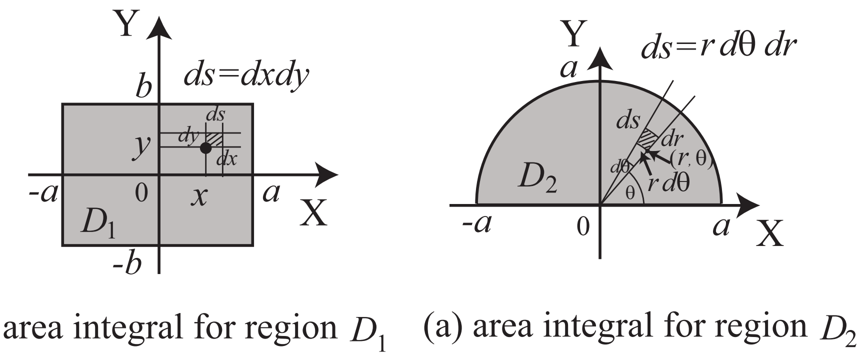


Figure 1.5: Examples of area integral of scalar functions (example of multiple integral)

$$\int_D x^2 ds = \int_D x^2 dxdy$$

where the region $D(D_1$ or $D_2)$ is shown in Figure 1.5.
Answer of Example 1.2:

$$\text{for } D_1 \ : \quad \int_{D_1} x^2 dxdy = \int_{-a}^{a} x^2 dx \int_{-b}^{b} dy = \frac{2}{3}a^3 \int_{-b}^{b} dy = \frac{4}{3}a^3 b$$

$$\text{for } D_2 \ : \quad \int_{D_2} x^2 dxdy = \int_{D_2} x^2 ds = \int_{D_2} r^2 \cos^2\theta r d\theta dr = \int_0^{\pi} \cos^2\theta d\theta \int_0^a r^3 dr$$
$$= \frac{a^4}{4}\int_0^{\pi} \frac{1+2\cos 2\theta}{2} d\theta = \frac{\pi a^4}{8}$$

(Change of integral variable from $x - y$ coordinate to polar coordinate)

2 Coordinate Systems

In mechanics, usual orthogonal curvilinear coordinate system (which is called **Cartesian coordinate system** or **Des Carte coordinate system**) is used. However another coordinate system is sometimes used depending on the actual problem. One of that is **Polar Coordinate System**.

2.1 Polar coordinate system (two dimension)

Polar coordinate expressions are useful for describing circular motions. The point vector $\boldsymbol{r}$ in the polar coordinate system is desribed by

$$\boldsymbol{r} = r\boldsymbol{e}_r \quad (r = |\boldsymbol{r}|) \tag{2.41}$$

where $\boldsymbol{e}_r$ is a unit (length =1) vector along the vector $\boldsymbol{e}_r$ directions as shown in Figure 2.6. Clearly from

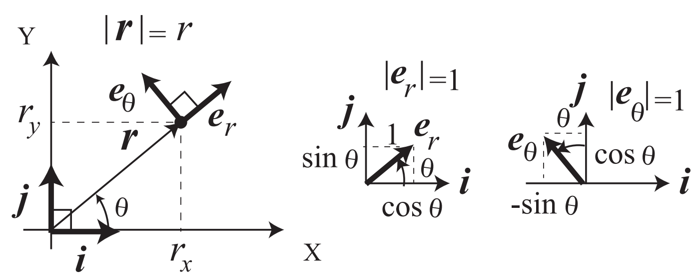


Figure 2.6: Polar coordinate system (two dimensional)

Figure 3, the relationship between the unit vector $\boldsymbol{i}, j$ in Cartesian coordinate and the unit vector $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ in Polar coordinate

$$\begin{cases} \boldsymbol{e}_r &= \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j} \\ \boldsymbol{e}_\theta &= -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j} \end{cases} \tag{2.42}$$

We now consider the velocity for the point vector $\boldsymbol{r}$. From Equation 2.41,

$$\dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\boldsymbol{e}}_r \tag{2.43}$$

From Equation 2.42, $\dot{\boldsymbol{e}}_r$ is desribed by

$$\dot{\boldsymbol{e}}_r = -\sin\theta\cdot\dot{\theta}\,\boldsymbol{i} + \cos\theta\cdot\dot{\boldsymbol{i}} + \cos\theta\cdot\dot{\theta}\,\boldsymbol{j} - \sin\theta\cdot\dot{\boldsymbol{j}} \tag{2.44}$$

where $\dot{\boldsymbol{i}} = \dot{\boldsymbol{j}} = 0$, thus

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\boldsymbol{e}_\theta \tag{2.45}$$

Similarly

$$\dot{\boldsymbol{e}}_\theta = -\cos\theta\cdot\dot{\theta}\,\boldsymbol{i} - \sin\theta\cdot\dot{\theta}\,\boldsymbol{j} = -\dot{\theta}\boldsymbol{e}_r \tag{2.46}$$

Therefore, the velocity and acceleration of $\boldsymbol{r}$ in the polar coordinate system is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \tag{2.47}$$

2.2 Cylindrical coordinate system (three dimension)

Cylindrical coordinate expressions are useful when describing geometrical motions along a cylinder. The point vector $\boldsymbol{p}$ in the cylindrical coordinate system is desribed by (see Figure 2.7)

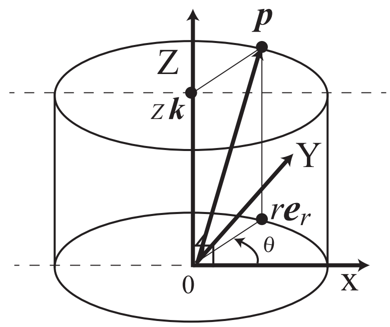


Figure 2.7: Cyrindrical coordinate system

$$\boldsymbol{p} = r\boldsymbol{e}_r + z\boldsymbol{k} \tag{2.48}$$

The X-Y plane of the cyrindrical coordinate frame is same as the polar coordinate system in the previous sub-section.

The velocity for the point p is

$$\dot{\boldsymbol{p}} = \boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + \dot{z}\boldsymbol{k} \tag{2.49}$$

2.3 Polar coordinate system (three dimension)

The three dimensional polar coordinate system are useful for describing spherical motions geometrically. The point vector $\boldsymbol{p}$ in the polar coordinate system is described by

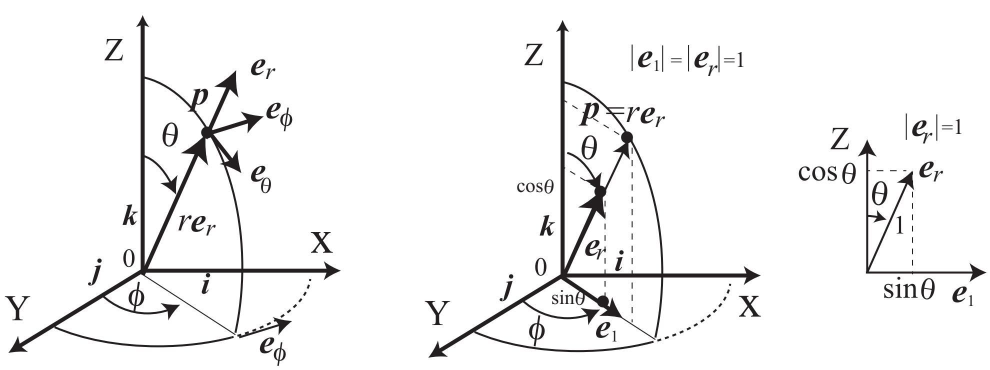


Figure 2.8: Three dimensional polar coordinate system

$$\boldsymbol{p} = r\boldsymbol{e}_r \tag{2.50}$$

From Figure 2.8,

$$\boldsymbol{e}_r = \sin\theta\boldsymbol{e}_1 + \cos\theta\boldsymbol{k} \tag{2.51}$$

$$\boldsymbol{e}_1 = \cos\phi\boldsymbol{i} + \sin\phi\boldsymbol{j} \tag{2.52}$$

Thus, $\boldsymbol{e}_r$ can be written using $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as

$$\boldsymbol{e}_r = \sin\theta\cos\phi\boldsymbol{i} + \sin\theta\sin\phi\boldsymbol{j} + \cos\theta\boldsymbol{k} \tag{2.53}$$

$\boldsymbol{e}_\theta$ can be also written using $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ as

$$\boldsymbol{e}_\theta = \cos\theta \boldsymbol{e}_1 - \sin\theta \boldsymbol{k} = \cos\theta\cos\phi \boldsymbol{i} + \cos\theta\sin\phi \boldsymbol{j} - \sin\theta \boldsymbol{k} \tag{2.54}$$

$\boldsymbol{e}_\phi$ is

$$\boldsymbol{e}_\phi = -\sin\phi \boldsymbol{i} + \cos\phi \boldsymbol{j} \tag{2.55}$$

By taking the derivatibe of $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$, we have

$$\left\{ \begin{array}{lcl} \dot{\boldsymbol{e}}_r & = & \dot{\theta}\boldsymbol{e}_\theta + \dot{\phi}\sin\theta \boldsymbol{e}_\phi \\ \dot{\boldsymbol{e}}_\theta & = & -\dot{\theta}\boldsymbol{e}_r + \dot{\phi}\cos\theta \boldsymbol{e}_\phi \\ \dot{\boldsymbol{e}}_\phi & = & -\dot{\phi}(\sin\theta \boldsymbol{e}_r + \cos\theta \boldsymbol{e}_\theta) \end{array} \right. \tag{2.56}$$

Using the relations, the position $\boldsymbol{p}$, the vector $\dot{\boldsymbol{p}}$ and the acceleration $\ddot{\boldsymbol{p}}$ can be written

$$\left\{ \begin{array}{lcl} \boldsymbol{p} & = & r\boldsymbol{e}_r \\ \dot{\boldsymbol{p}} = \boldsymbol{v} & = & \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta + r\dot{\phi}\sin\theta \boldsymbol{e}_\phi \\ \ddot{\boldsymbol{p}} = \boldsymbol{a} & = & (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\boldsymbol{e}_\theta \\ & & +\frac{1}{r\sin\theta}\frac{d}{dt}(r^2\dot{\phi}\sin^2\theta)\boldsymbol{e}_\phi \end{array} \right. \tag{2.57}$$

3 Newton's Motion of Equation (Physical Law of Motions)

Newton's law of motion is a physical law that governs the motion of all objects in the world of classical mechanics. Note that the physical law is not a definition nor thorem in mathematics, but a experimental law. Newton's law consists of three laws: the law of inertia, the law of motion, and the law of action and reaction.

3.1 Law of inertia (first law)

The law of inertia is stated as

Law of inertia (first law);
when a point mass is not affected by any forces, it remains stationary or moves at a constant speed.

In many physical phenomena on the earth, this law cannot be confirmed due to friction of air (discussed later), but it is confirmed that this law is correct in outer space. A coordinate system in which the law of inertia holds is called an **inertial coordinate system**. The law of inertia can be rephrased as a law that recognizes the existence of an inertial coordinate system.

3.2 Law of motion (second law)

We first define the **momentum** $\boldsymbol{p}$ by the following equation

$$\boldsymbol{p} \triangleq m\boldsymbol{v} \tag{3.58}$$

where m is a point mass and $\boldsymbol{v}$ is the velocity for the point mass. The second law of motion (or just law of motion) can be stated as "when a force $\boldsymbol{F}$ is applied, the law of inertia is not satisfied, and the speed changes. The force is equal to the change in momentum over time". That is

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} \tag{3.59}$$

which is called Newton's equation of motion, or just called law of motion. When mass m is not changes on time t, then

Law of motion (second law)=Newton's equation of motion

$$\frac{d\boldsymbol{p}}{dt} = \frac{d(m\boldsymbol{v})}{dt} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a} = \boldsymbol{F} \tag{3.60}$$

This equation is also called Newton's equation of motion.

3.3 Law of action and reaction force (third law)

When pushing someting such as wall with force F_1 and the pushing point does not move (or moves with constant speed), then reaction force F_2 is pushing back with oposite direction.

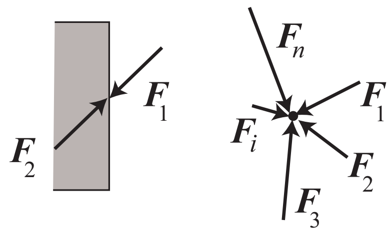


Figure 3.9: Law of action and reaction force

$$\boldsymbol{F}_1 = -\boldsymbol{F}_2 \tag{3.61}$$

More generally, when a point does not move (or moves with constant speed), multiple forces (force vectors) are balanced at the point.

$$\sum_{i=1}^{n} \boldsymbol{F}_i = 0 \quad \text{or} \quad \sum_{i=1}^{n} F_{ix} = 0, \sum_{i=1}^{n} F_{iy} = 0, \sum_{i=1}^{n} F_{iz} = 0, \tag{3.62}$$

Note that the force (and the moment) including momentum follows vector characteristics in the mathematical meaning. That means the forces can be decomposed and composed.

3.4 Motion of equation in Cartesian (Des Carte) coordinate system

The motion of equation is described by Equation 3.59 or 3.60. The expression of the motion of equation is, however, different depending on the coordinate system, because the expression of acceleration $\boldsymbol{a}$ is different depending on the coordinate system. The acceleration $\boldsymbol{a}$ in Cartesian (Des Carte) coordinate system is

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k} = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{x}\boldsymbol{k} \tag{3.63}$$

The force $\boldsymbol{F}$ in Cartesian (Des Carte) coordinate system is also described by

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k} \tag{3.64}$$

Thus, the elements of motion of equation in Cartesian (Des Carte) coordinate system is

$$\left\{ \begin{array}{rcl} m\ddot{x} & = & F_x \\ m\ddot{y} & = & F_y \\ m\ddot{z} & = & F_z \end{array} \right. \tag{3.65}$$

3.5 Motion of equation in polar coordinate system (two dimension)

The acceleration $\boldsymbol{a}$ in polar coordinate system is obtained by taking derivative for Equation 2.47 as

$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta})\boldsymbol{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_\theta \tag{3.66}$$

Thus, the elements of motion of equation in polar coordinate system is

$$\left\{ \begin{array}{rcl} m(\ddot{r} - r\dot{\theta}) & = & F_r \\ m(2\dot{r}\dot{\theta} + r\ddot{\theta}) & = & F_\theta \end{array} \right. \tag{3.67}$$

3.6 Solution of Motion of equation (uniform gravitational field)

By solving the motion of equation, we get to know any motions for specific problems. Solving the motion of equation means solving the differential equation of the motion of equation. Which means integrating the motion of equation.

For example, the motion of a mass point in gravitational force field as in Figure 3.10 is obtained by integrating

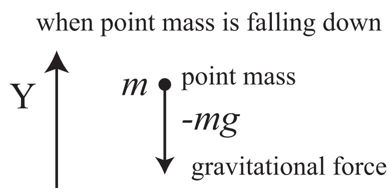


Figure 3.10: Motion in uniform gravitational field

$$\begin{cases} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg \\ m\ddot{z} &= 0 \end{cases} \tag{3.68}$$

when taking the Y direction for the oposite of gravitational direction. The Y direction velocity v_y is

$$\dot{y} = v_y = -gt + C_1 \tag{3.69}$$

The position is

$$y = -\frac{1}{2}gt^2 + C_1 t + C_2 \tag{3.70}$$

The integral constants C_1, C_2 are determined by initial or terminal consditions.

3.7 Parabolic movement in uniform gravitational field

The falling body performs a parabolic movement in uniform gravitational field. Here we confirm the movement using the motion of equation.

As shown in the figure, consider the case where a point mass is thrown at an initial velocity v_0 in a direction of angle α for the horizontal direction. The equation of motion is

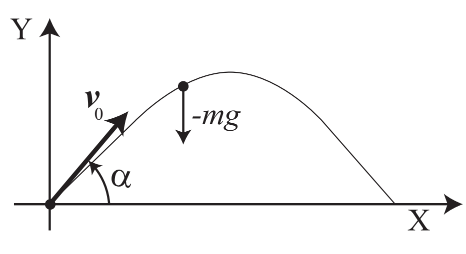


Figure 3.11: Parabolic movement in uniform gravitational field

$$\begin{cases} (\rightarrow) & m\ddot{x} &= 0 \\ (\uparrow) & m\ddot{y} &= -mg \end{cases} \tag{3.71}$$

Solving it with initial condition at time $t = 0$, $v_x = v_0 \cos\alpha, v_y = v_0 \sin\alpha$, $x = 0, y = 0$, then the velocity is

$$\begin{cases} \dot{x} &= v_x &= v_0 \cos\alpha \\ \dot{y} &= v_y &= -gt + v_0 \sin\alpha \end{cases} \tag{3.72}$$

The position (x, y) is

$$\begin{cases} x = v_0 t \cos\alpha \\ y = -\frac{1}{2}gt^2 + v_0 t \sin\alpha \end{cases} \tag{3.73}$$

3.8 Movement of falling object with air friction

When an object is falling down in air with air friction where the friction force typically written $b\dot{y} = bv$, then motion of equation is

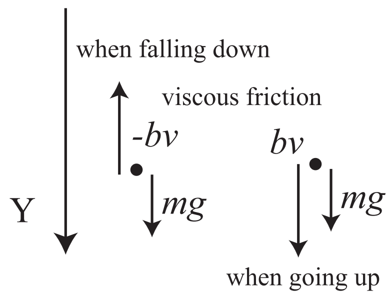


Figure 3.12: Falling object with air friction

$$(\downarrow) m\ddot{y} = mg - b\dot{y} \tag{3.74}$$

where b is called viscous friction of air.

When the point mass is going up, it affects the friction force downward, thus

$$(\downarrow) m\ddot{y} = mg + b\dot{y} \tag{3.75}$$

Now we solve some problems using the motion of equations. When an objevt with mass m is falling with initial ($t = 0$) condition $v = \dot{y} = 0$, then using Equation 3.74 and writing $v = \dot{y}$ leads

$$\dot{v} + \frac{b}{m}v = g \tag{3.76}$$

Solving the differential equation considering Equations 1.11 and 1.13,

$$v = Ce^{-\frac{b}{m}t} + \bar{v} \tag{3.77}$$

where $\bar{v}$ is a special solution for Equation 3.66, which is $\bar{v} = \frac{mg}{b}$. By substituting the initial conditions $v = 0$ at time $t = 0$ and using it

$$v = \frac{mg}{b}(1 - e^{-\frac{b}{m}t}) \tag{3.78}$$

It can be seen that the speed v approaches a constant value $v = \frac{mg}{b}$ (this is called **final velocity**) after a sufficient time.

(Problem 3.1)
What happened when an object is thrown up for negative Y direction with initial velocity $-v_0$?

3.9 Two types of friction on a surface

When an object moves on a surface, it receives frictional force from the surface. Mainly for this reason, even if it starts moving at the initial speed v, it stops immediately. It is known that there are roughly two types of the friction force. One is called **viscous friction force**, and its physical law is described by the following equation.

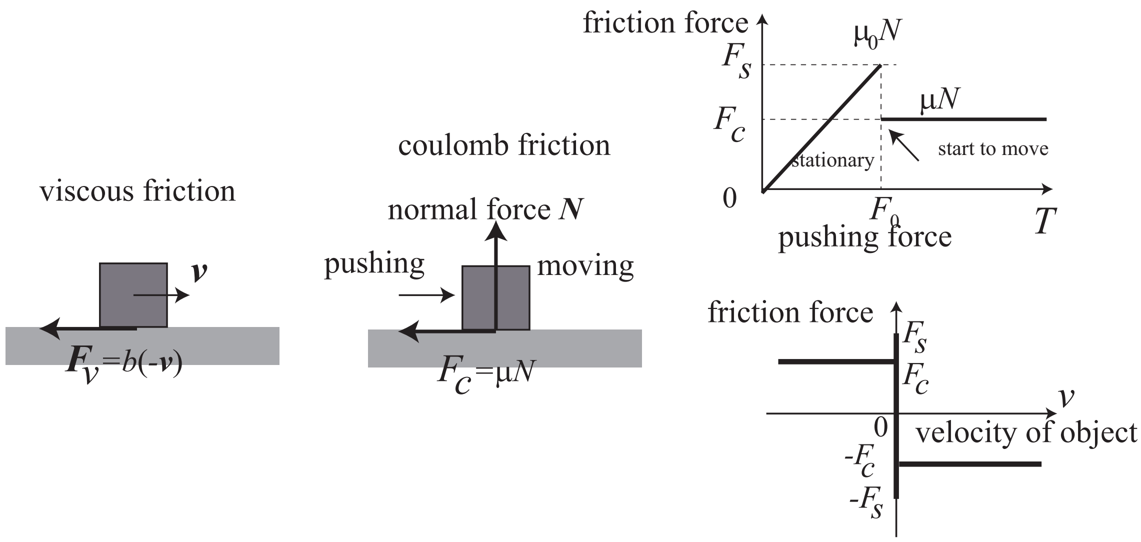


Figure 3.13: Viscous friction and coulomb friction force on a surface

$$\boldsymbol{F}_v = b(-\boldsymbol{v}) \tag{3.79}$$

where b is viscous friction coefficient. This force acts in the opposite direction to the velocity vector of the object. The friction by air in the former subsection is the same viscous friction.

Another is called **dynamic (kinematic) friction force**. The friction law is also called **Amonton's law of friction** or **coulomb friction law**. Which is described

$$\boldsymbol{F}_c = -\frac{\boldsymbol{v}}{|\boldsymbol{v}|}\mu N \tag{3.80}$$

where N is normal force for the surface. Note that it never depends on the magnitude of the velocity v, nor the area as it looks. It only depends the load mg (or more precisely N). When the pushing velocity is changed, the coulomb friction force takes constant $-F_c$ for the positive velocity v. Note that when the pressing force gradually increases from the stationary state, the pressing force becomes the static friction force as it is.. The object starts to move at the moment when the maximum static friction force ($F_s = \mu_0 R$) is reached, and rapidly decreases to a constant dynamic friction force (see Figure 3.13). Where μ is called static friction coefficient and μ_0 is called maximum static friction coefficient.

3.10 Constrained motion on a slope

We here consider the situation that an object slides down a slope as in Figure 3.14. The equations of

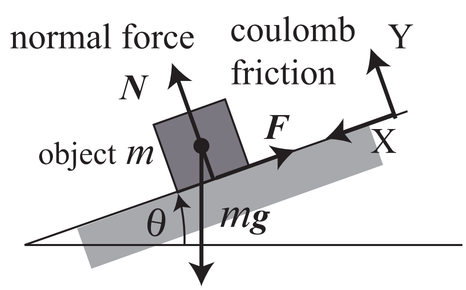


Figure 3.14: Falling on a slope with coulomb friction

motion for X direction and Y direction are

$$\begin{cases} (\swarrow) m\ddot{\boldsymbol{x}} &= mg\sin\theta - F \\ (\nwarrow) m\ddot{\boldsymbol{y}} &= -mg\cos\theta + R \end{cases} \tag{3.81}$$

where F is coulomb friction force and $R = mg\cos\theta$ (because $\ddot{y} = 0$) is normal force. The object is stationary until the X-direction component of gravity force reaches the maximum static friction force when the slope angle is increased. The condition is

$$mg\sin\theta = -\mu_0 R = \mu_0 mg\cos\theta \tag{3.82}$$

which becomes $\tan\theta = \mu_0$ where the θ is called **friction angle**. Once the object starts to move beyond the maximum static friction, the first Equation of 3.81 is

$$m\ddot{x} = mg\sin\theta - \mu mg\cos\theta \tag{3.83}$$

which is $\ddot{x} = g(\sin\theta - \mu\cos\theta)$. Solving the differential equation leads

$$x(t) = \frac{g}{2}t^2(\sin\theta - \mu\cos\theta) + C_1 t + C_2 \tag{3.84}$$

3.11 Constrained motion with circular shape (simple pendulum)

We here consider the situation that an object is constrained with a rope in a vertical plane (simple pendulum) as in Figure 3.15. The motion of equation in a polar coordinate is

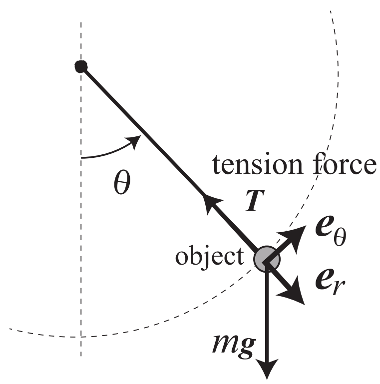


Figure 3.15: Constrained motion with circular shape

$$\begin{cases} r\text{-direction } (\searrow) & m(\ddot{r} - r\dot{\theta}^2) &= mg\cos\theta - T \\ \theta\text{-direction } (\nearrow) & m(2\dot{r}\dot{\theta} + r\ddot{\theta}) &= -mg\sin\theta \end{cases} \tag{3.85}$$

Because of $\ddot{r} = \dot{r} = 0$ and $r = l$, and by assuming $\theta \approx 0$, the second equation is

$$\ddot{\theta} = -\frac{g}{l}\theta$$

This result in simple vibration solution (see section 1). Thus the solution can be written

$$\theta(t) = A\cos(\omega t + \alpha) = A(\cos\sqrt{\frac{g}{l}}t + \alpha) \tag{3.86}$$

When the case of $\theta = \pi/6$ and $\dot{\theta} = 0$ at time $t = 0$, then

$$\theta(t) = \frac{\pi}{6}\cos\sqrt{\frac{g}{l}}t$$

4 Vibration

When the characteristic equation of second order differential equation has complex solution including pure imaginary solution, the corresponding motion becomes **vibration** one. In this section we learn about the vibration solutions.

4.1 Simple vibration

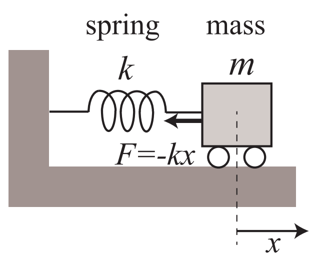


Figure 4.16: Spring mass system

The motion of equation for the mass-spring system in Figure 4.16 is

$$(\rightarrow) m\ddot{x} = -kx \tag{4.87}$$

This becomes a second order differential equation $m\ddot{x} + kx = 0$ or

$$\ddot{x} + \omega^2 x = 0$$

As we have already seen, the solution becomes a simple vibration as

$$x(t) = A\cos(\omega t + \alpha)$$

The period of the oscillation T is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} \tag{4.88}$$

where ω is called natural angular frequency. Now we understand the motion of simple pendulum is also a simple vibration.

4.2 Damped vibration

We here consider the motion when the mass part is affected by viscous friction as in Figure 4.17 The motion of equation is

$$(\rightarrow) m\ddot{x} = -kx - b\dot{x} \tag{4.89}$$

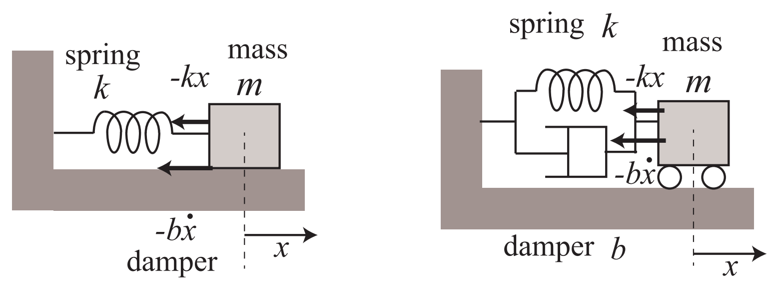


Figure 4.17: Damped spring mass system

We can write the differential equation as

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0 \quad (2\beta = \frac{b}{m},\ \omega_0 = \sqrt{\frac{k}{m}}) \tag{4.90}$$

Characteristic equation is

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \tag{4.91}$$

The root is

$$\lambda = -\beta \pm \sqrt{\beta^2 - \omega_0^2} \tag{4.92}$$

We now should check the equation according to the real, duplicate complex solutions.

[strong damp (over damping)] (case of $\beta > \omega_0$)
For this case the solutions of λ are two different real ones which we write $\lambda = \lambda_1, \lambda_2$, then the solution is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \tag{4.93}$$

[critical damping] (case of $\beta = \omega_0$)
For this case, the characteristic equation is

$$\lambda^2 + 2\beta\lambda + \beta^2 = (\lambda + \beta)^2 = 0$$

Thus the solution of λ is one real duplicate solution which we write $\lambda = -\beta$, then the solution is

$$x(t) = (C_1 + C_2 t)e^{-\beta t} \tag{4.94}$$

[weak damp (damped oscillation)] (case of $\beta < \omega_0$)
For this case the solutions of λ are two complex solutions which we write $\lambda = -\beta \pm i\sigma \quad (\sigma = \sqrt{\omega_0^2 - \beta^2})$, then the solution is

$$x(t) = C_1 e^{(-\beta+\sigma i)t} + C_2 e^{(-\beta-\sigma i)t} = C_1 e^{-\beta t} \times e^{\sigma i t} + C_2 e^{-\beta t} \times e^{-\sigma i t} = e^{-\beta t}(C_1 e^{\sigma i t} + C_2 e^{-\sigma i t}) \tag{4.95}$$

By applying Euler's theorem ($e^{i\theta} = \cos\theta + i\sin\theta$),

$$x(t) = e^{-\beta t}(C_1(\cos\sigma t + i\sin\sigma t) + C_2(\cos\sigma t - i\sin\sigma t)) = e^{-\beta t}(A\cos\sigma t + B\sin\sigma t) = De^{-\beta t}\sin(\sigma t + \alpha) \tag{4.96}$$

This means damped oscillation.

5 Work and Energy

In this section, we learn the idea of work and energy based on force and moving distance. The work is basically identical with the idea of energy which is derived from equation of motion.

5.1 Definition of work

We here consider that an object is pushed by an added constant force $\boldsymbol{F}$ and it moves with length $|\boldsymbol{s}|$ along the direction of $\boldsymbol{s}$ as in Figure 5.18 Then the work W done by the force is defined by

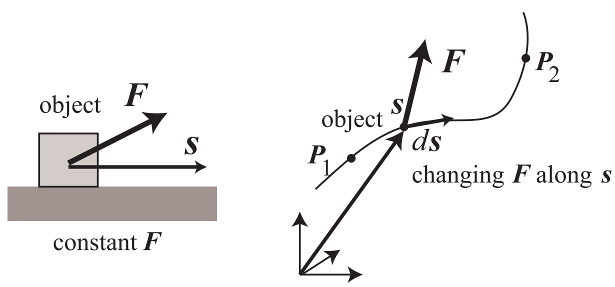


Figure 5.18: Definition of work

$$W \triangleq \boldsymbol{F} \cdot \boldsymbol{s} \tag{5.97}$$

Note that the calculation is inner product, thus the work W has scalar value. When the object moves along a curve of $\boldsymbol{s}$ with added changing force $\boldsymbol{F}$

$$W \triangleq \int_{P_1}^{P_2} \boldsymbol{F} \cdot d\boldsymbol{s} \tag{5.98}$$

The force $\boldsymbol{F} = (F_x, F_y, F_z)$ and $d\boldsymbol{s} = (dx, dy, dz)$, thus the work can be calculated by

$$W = \int_{P_1}^{P_2} F_x dx + F_y dy + F_z dz = \int_{x_1}^{x_2} F_x dx + \int_{y_1}^{y_2} F_y dy + \int_{z_1}^{z_2} F_z dz \tag{5.99}$$

This integral is called **line integral**.

(Problem 5.1)
Calculate the work when an object is moved along the curve $C_1(P_1 \to P_2 \to P_3)$ and $C_2(P_1 \to P_3)$ when the added force is given by $\boldsymbol{F} = (x - y)\boldsymbol{i} + (ax)\boldsymbol{j}$ in the Figure 5.19. The work along C_1 is

$$X_{C_1} = W_{P_1 \to P_2} + W_{P_2 \to P_3}$$

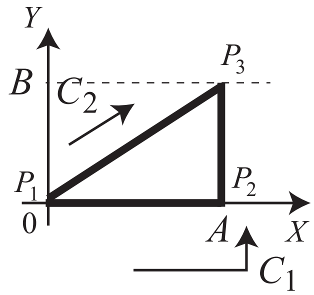


Figure 5.19: Work along C_1 and C_2

$$W_{P_1 \to P_2} = \int_{P_1}^{P_2} F_x dx + F_y dy = \int_0^A F_x dx = \int_0^A (x-y)dx = \left[\frac{x^2}{2}\right]_0^A = \frac{A^2}{2}$$
$$W_{P_2 \to P_3} = \int_{P_2}^{P_3} F_x dx + F_y dy = \int_0^B F_y dy = \int_0^B axdy = [aAy]_0^B = aAB$$

On the other hand, the work along C_2 is

$$W_{C_2} = W_{P_1 \to P_3} = \int_{P_1}^{P_3} F_x dx + F_y dy = \int_0^A (x-y)dx + \int_0^B axdy$$
$$= \int_0^A (x - \frac{B}{A}x)dx + \int_0^B a\frac{Ay}{B}dy = \frac{A}{2}(A-B) + \frac{aAB}{2}$$

When $a = -1$ two works along C_1 and C_2 are same. However, in general, the works for different routes are not same. Thus the work depends on the route.

5.2 Conservative force and potential

The work generally depends on the path of the motion. However, there are special forces whose work does not depend on its path. This subsection discuss such special force called conservative force.

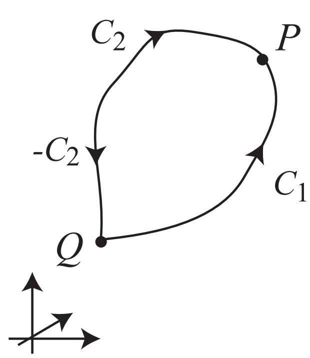


Figure 5.20: The work does not depend on its path

When the works along the path C_1 and C_2 are same as Figure 5.20, then we can write

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s} \tag{5.100}$$

By using the fact $\int_{C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{-C_2} \boldsymbol{F} \cdot d\boldsymbol{s}$,

$$\int_{C_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{-C_2} \boldsymbol{F} \cdot d\boldsymbol{s} = 0 \tag{5.101}$$

This integral means integrating along arbitrary closed curve C, and can be written as follows.

$$\oint_C \boldsymbol{F} \cdot d\boldsymbol{s} = 0 \tag{5.102}$$

The force satisfied the equation for any curve C is called **conservative force**.

When the force is the conservative force, the work W is determined only for the terminal point $P(x, y, z)$ and $Q(x_0, y_0, z_0)$ thus

$$W = \int_Q^P \boldsymbol{F} \cdot d\boldsymbol{s} = -[U(x, y, z) - U(x_0, y_0, z_0)] \tag{5.103}$$

When we take a reference point as Q and set $U(Q) = U(x_0, y_0, z_0) = 0$ then

$$U(x, y, z) = -\int_Q^P \boldsymbol{F} \cdot d\boldsymbol{s} \tag{5.104}$$

This integral $U(x, y, z)$ is called **Potential (energy)**.

We next consider about the condition to judge that the given force is conservative force or not. The small deviation of the potential ΔU which is the potential at a little bit different point $P'(x + \Delta x, y, z)$ from the original point $P(x, y, z)$, then

$$\Delta U = U(x, y, z) - U((x + \Delta x, y, z) = -\int_{P'}^P \boldsymbol{F} \cdot d\boldsymbol{s}$$

$$= -\int_{(x+\Delta x, y, z)}^{(x,y,z)} F_x dx + F_y dy + F_z dz = -\int_{x+\Delta x}^{x} F_x dx = -F_x \Delta x \tag{5.105}$$

By taking the limit of $\Delta x \to 0$,

$$F_x = \lim_{\Delta x \to 0} \frac{\Delta U}{\Delta x} = -\frac{\partial U}{\partial x} \tag{5.106}$$

Similarly for F_y and F_z,

$$F_y = -\frac{\partial U}{\partial y}, \quad F_z = -\frac{\partial U}{\partial z} \tag{5.107}$$

These equation can be written simply

$$\boldsymbol{F} = -(\frac{\partial U}{\partial x}\boldsymbol{i} + \frac{\partial U}{\partial y}\boldsymbol{j} + \frac{\partial U}{\partial z}\boldsymbol{k}) = -\nabla U \tag{5.108}$$

The symbol ∇ is called **nabla** means gradient (slope) of something (potential U in this case).

By taking one more partial derivative of $F_x = -\frac{\partial U}{\partial x}$ on y and $F_y = -\frac{\partial U}{\partial y}$ on x

$$\frac{\partial F_x}{\partial y} = -\frac{\partial^2 U}{\partial x \partial y}, \quad \frac{\partial F_y}{\partial x} = -\frac{\partial^2 U}{\partial y \partial x} \tag{5.109}$$

Since the second-order partial derivative does not depend on its order,

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x} \tag{5.110}$$

Similarly

$$\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}, \quad \frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z} \tag{5.111}$$

These are the equivalent conditions of conservative force for $\boldsymbol{F} = (F_x, F_y, F_z)$. When the force is satisfied with these conditions, then the force is called **conservative force**.

5.3 From the motion of equation to kinetic energy

The motion of equation is

$$m\frac{d\boldsymbol{v}}{dt} = \boldsymbol{F}$$

by multiplying $\boldsymbol{v}$ with inner product for each side,

$$m\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} = \boldsymbol{F} \cdot \boldsymbol{v} \tag{5.112}$$

The left part is

$$\frac{d\boldsymbol{v}}{dt} \cdot \boldsymbol{v} = (\frac{dv_x}{dt}, \frac{dv_y}{dt}, \frac{dv_z}{dt}) \cdot (v_x, v_y, v_z) = v_x\frac{dv_x}{dt} + v_y\frac{dv_y}{dt} + v_z\frac{dv_z}{dt} = \frac{1}{2}\frac{d}{dt}(v_x^2 + v_y^2 + v_z^2) \tag{5.113}$$

Thus

$$m\frac{d}{dt}(\frac{1}{2}v^2) = \boldsymbol{F} \cdot \boldsymbol{v} \tag{5.114}$$

where $v = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}}$ is the magnitude of the velocity $\boldsymbol{v}$. By integrating for both side on t,

$$m\int \frac{d}{dt}(\frac{1}{2}v^2)dt = \int \boldsymbol{F} \cdot \boldsymbol{v}dt \tag{5.115}$$

This integral is called **energy integral**. By calculating definite integral and using $\boldsymbol{v} = \dfrac{d\boldsymbol{s}}{dt}$

$$m\left[\frac{1}{2}v^2\right]_1^2 = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}dt = \int_{P_1}^{P_2} \boldsymbol{F} \cdot d\boldsymbol{s} = W \tag{5.116}$$

The left part $\frac{1}{2}mv^2$ is called **kinetic energy**. The right part is the work, thus it means **work** by added force $\boldsymbol{F}$. Therefore this result means **kinetic energy is equivalent with work**. This kind of calculation is called "**Energy Integration**". Another energy integration leads to another (similar) energy conservation law in the next subsection.

5.4 Energy conservation of kinetic energy and potential energy

When an point mass is in the gravitational field,

$$m\ddot{\boldsymbol{r}} = -mg$$

which is

$$m\dot{\boldsymbol{v}} = -mg \tag{5.117}$$

By taking energy integral as in the previous subsection and moving the point mass m from h_1 to h_2 in a vertical direction

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = -mg\int_{h_1}^{h_2} d\boldsymbol{s} = -mg\int_{h_1}^{h_2} dy = mgh_1 - mgh_2 \tag{5.118}$$

Or

$$\frac{1}{2}mv_1^2 + mgh_1 = \frac{1}{2}mv_2^2 + mgh_2$$

This equation means a conservation law for the sum of kinetic energy and potential energy. This is because the gravitational force mg is a conservation force (see left of Figure 5.21).

(Problem 5.1)

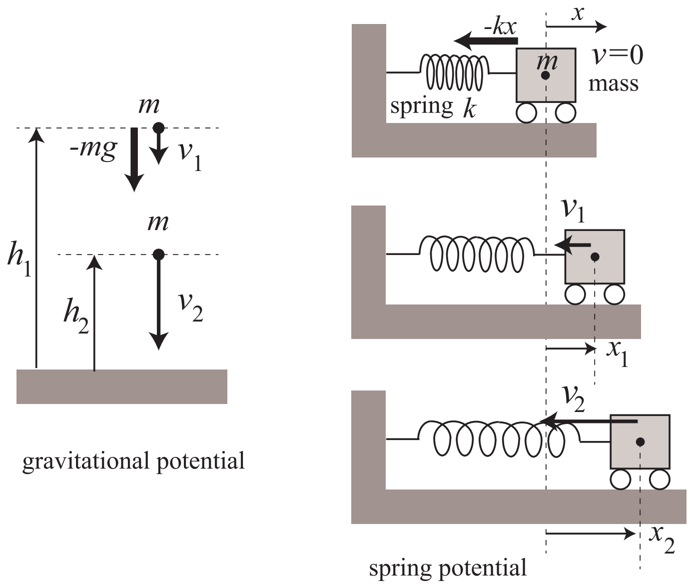


Figure 5.21: Energy conservation of kinetic energy and potential energy

Prove the gravitational force $-mg$ is a conservation force.

When a mass connected with a spring is moving,

$$m\ddot{\boldsymbol{x}} = -kx$$

which is

$$m\dot{\boldsymbol{v}} = -kx \tag{5.119}$$

By taking energy integral as in the previous subsection and considering the points x_1 and x_2

$$\frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = -k\int_{x_1}^{x_2} xd\boldsymbol{s} = -k\int_{x_1}^{x_2} xdx = \frac{1}{2}kx_1^2 - \frac{1}{2}kx_2^2 \tag{5.120}$$

Or

$$\frac{1}{2}mv_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}mv_2^2 + \frac{1}{2}kx_2^2$$

This equation means a conservation law for the sum of kinetic energy and potential energy of spring. This is because the spring force kx is also a conservation force (see right of Figure 5.21).

6 Appendix

[A] Proof of Euler's theorem

We define a function $y(x)$ as

$$y(x) = \cos x + i \sin x \tag{6.121}$$

By taking the differential for the both side on x

$$\frac{dy}{dx} = -\sin x + i \cos x = i(\cos x + i \sin x) = iy \tag{6.122}$$

This is a first order differential equation. The solution can be written from Equation 1.13 as

$$y = Ce^{ix} \tag{6.123}$$

Because $y(0) = 1$, $C = 1$. Thus

$$\cos x + i \sin x = e^{ix} \tag{6.124}$$

This is called **Euler's theorem**.