ROBOTICS

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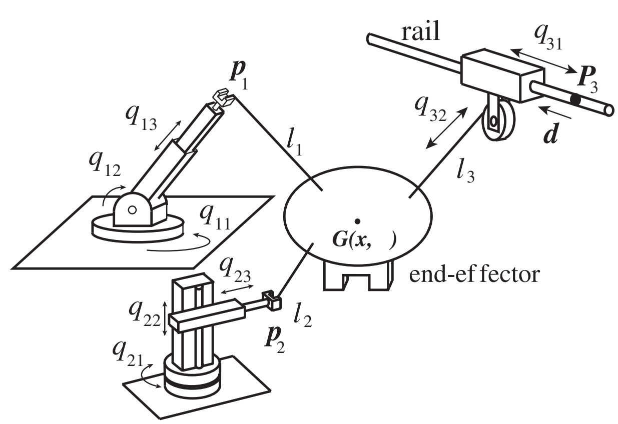
Ver. 3.1

Motoji Yamamoto



Chapter 1

KINEMATICS

1.1 Definition of Rotation Matrix

When we control a robot to execute a given task, the motion of the robot should be described mathematically with some manners. The representation of the motion includes positions and orientations of the robot hand and each part of the robot. To represent the orientation, we first introduce "**rotation matrix**" R. We now have two coordinate frames Σ_A and Σ_B. (See Fig. 1.1.) The Σ_A represents a reference coordinate frame as shown in Fig. 1.2. Then the unit vectors ${}^A\boldsymbol{x}_B, {}^A\boldsymbol{y}_B, {}^A\boldsymbol{z}_B$ in the Σ_A coordinate frames are defined as

$$\begin{aligned} {}^A\boldsymbol{x}_B &: \text{unit vector along } X_B \text{ in } \Sigma_A \text{ coordinate frame} \\ {}^A\boldsymbol{y}_B &: \text{unit vector along } Y_B \text{ in } \Sigma_A \text{ coordinate frame} \\ {}^A\boldsymbol{z}_B &: \text{unit vector along } Z_B \text{ in } \Sigma_A \text{ coordinate frame} \end{aligned}$$

In this text book, the left upper subscript of a vector indicates the coordinate frame where the vector is described in the coordinated frame. We now define the "rotation matrix" ${}^AR_B$ by

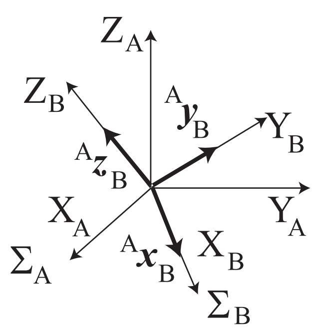


Fig. 1.1 Coordinate frames Σ_A and Σ_B

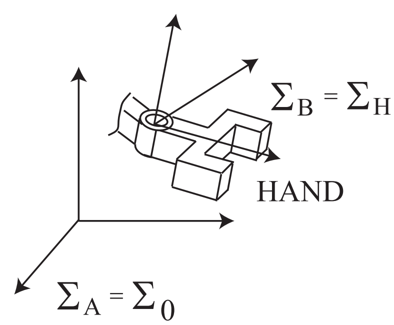


Fig. 1.2 Orientation of hand

$${}^AR_B \triangleq [{}^A\boldsymbol{x}_B \mid {}^A\boldsymbol{y}_B \mid {}^A\boldsymbol{z}_B] \quad \text{where} \quad {}^A\boldsymbol{x}_B = \begin{bmatrix} {}^Ax_{Bx} \\ {}^Ax_{By} \\ {}^Ax_{Bz} \end{bmatrix} \tag{1.1}$$

The rotation matrix represents an orientation of the coordinate frame Σ_B with reference to the Σ_A coordinate frame. When a hand is fixed with Σ_B coordinate frame as in Fig. 1.2, then the rotation matrix ${}^AR_B$ represents orientation of the hand with reference to the Σ_A coordinate frame.

1.2 Coordinate Transformation of Vector

We here define a vector $\boldsymbol{r}_0$ in two coordinate frames Σ_A and Σ_B, as

$$^A\boldsymbol{r}_0 : \text{vector } \boldsymbol{r}_0 \text{ in } \Sigma_A$$
$$^B\boldsymbol{r}_0 : \text{vector } \boldsymbol{r}_0 \text{ in } \Sigma_B$$

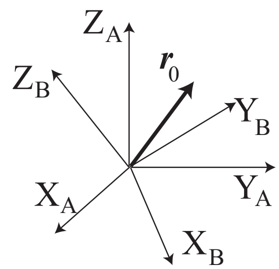


Fig. 1.3 Vector r_0 in two coordinate frames Σ_A and Σ_B

Note that $^A\boldsymbol{r}_0 \neq {}^B\boldsymbol{r}_0$, although the two vectors represent same point. The vector $^B\boldsymbol{r}_0 = [^Br_{0x}, {}^Br_{0y}, {}^Br_{0z}]^T$ is represented by the form of

$$^B\boldsymbol{r}_0 = {}^Br_{0x}\boldsymbol{i} + {}^Br_{0y}\boldsymbol{j} + {}^Br_{0z}\boldsymbol{k}$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ is the unit vectors along each X_B, Y_B, Z_B axis. When we change the reference coordinate from Σ_B to Σ_A, then the vector is

$$^A\boldsymbol{r}_0 = {}^Br_{0x}{}^A\boldsymbol{x}_B + {}^Br_{0y}{}^A\boldsymbol{y}_B + {}^Br_{0z}{}^A\boldsymbol{z}_B$$

Then we can represent the vector $^A\boldsymbol{r}_0$ using ^AR_B and $^B\boldsymbol{r}_0$,

$$^A\boldsymbol{r}_0 = {}^AR_B{}^B\boldsymbol{r}_0 \tag{1.2}$$

We easily derive the following formula of rotation matrix from the definition.

$$(^AR_B)^{-1} = (^AR_B)^T = {}^BR_A \tag{1.3}$$

$$^AR_B{}^BR_C = {}^AR_C \tag{1.4}$$

Followings are special cases of rotation matrices.

$$\text{rotate } \theta \text{ about Z-axis} \quad R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} = Rot(Z, \theta) \tag{1.5}$$

$$\text{rotate } \theta \text{ about Y-axis} \quad R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} = Rot(Y, \theta) \tag{1.6}$$

$$\text{rotate } \theta \text{ about X-axis} \quad R_X(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} = Rot(X, \theta) \tag{1.7}$$

1.3 Euler Angles and Rotation Matrix

As an another way to describe the orientation of rigid object such as robotic hand in three dimensional space, Euler angles (parameters) are often used. A common definition of Euler angles using the rotation matrix is

[Step 1] Rotate ϕ about Z_0 $\quad {}^0R_{0'} = Rot(Z, \phi)$
[Step 2] Rotate θ about $Y_{0'}$ $\quad {}^{0'}R_{0''} = Rot(Y, \theta)$
[Step 3] Rotate ψ about $Z_{0''}$ $\quad {}^{0''}R_A = Rot(Z, \psi)$

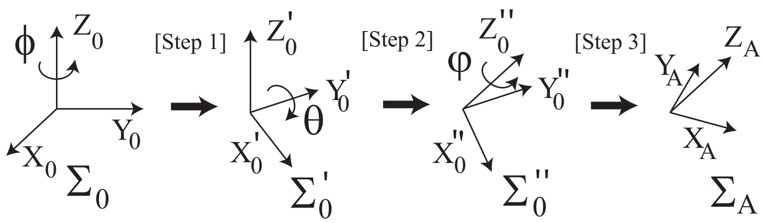


Fig. 1.4 Euler angles

then the rotation matrix representing Euler angles ${}^0R_A$ is

$$
{}^0R_A = {}^0R_{0'}{}^{0'}R_{0''}{}^{0''}R_A = \begin{bmatrix} C_\phi C_\theta C_\psi - S_\phi S_\psi & -C_\phi C_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta \\ S_\phi C_\theta C_\psi + C_\phi S_\psi & -S_\phi C_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta \\ -S_\theta C_\psi & S_\theta S_\psi & C_\theta \end{bmatrix} \tag{1.8}
$$

where $C_x = \cos x,\ S_x = \sin x$.
Note that changing the order of the transformation leads to another definition of ${}^0R_A$. Actually another definitions of the order is also used. For example $Z \Rightarrow X \Rightarrow Z$ or $Y \Rightarrow X \Rightarrow Y$.

[**Find Euler angles for given orientation of hand (Direct Method)**]

By tracing back the definition of Euler angles,

[Step 1]	Rotate $-\psi$ about Z_A axis	until Y_A is on X-Y plane of Σ_0 (Generally we get two solutions for ψ.)
[Step 2]	Rotate $-\theta$ about Y'' axis	until X'' is on X-Y plane of Σ_0 and Z'' comes $Z_0(= Z')$
[Step 3]	Rotate $-\phi$ about Z' axis	until X' is X of Σ_0

[**Find Euler angles for given rotation matrix (Calculation using the elements of** R**)**]

At first we find the elements of rotation matrix by the definition

$$
R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix},
$$

then we can calculate Euler angles by the elements of R by

$$
\begin{cases} \theta = \text{atan2}(\pm\sqrt{R_{13}^2 + R_{23}^2}, R_{33}) \\ \phi = \text{atan2}(\frac{R_{23}}{S_\theta}, \frac{R_{13}}{S_\theta}) \\ \psi = \text{atan2}(\frac{R_{32}}{S_\theta}, -\frac{R_{31}}{S_\theta}) \end{cases} \qquad (S_\theta \neq 0) \tag{1.9}
$$

where atan2$(Y, X) = \tan^{-1}(\frac{Y}{X})$. Note that the duplex symbol means two sets of solutions.

When $S_\theta = 0$,

$$
\begin{cases} \psi = \text{arbitrary} \\ \theta = 0\ (C_\theta = 1), \quad \phi = \text{atan2}(R_{21}, R_{22}) - \psi \\ \theta = \pi\ (C_\theta = -1), \quad \phi = -\text{atan2}(R_{21}, R_{22}) + \psi \end{cases}
$$

1.4 Definition of Roll, Pitch, Yaw Angles

The roll pitch yaw angles are defined by

[Step 1]	Rotate ϕ about Z_0	${}^0R_{0'} = Rot(Z, \phi)$
[Step 2]	Rotate θ about $Y_{0'}$	${}^{0'}R_{0''} = Rot(Y, \theta)$
[Step 3]	Rotate ψ about $X_{0''}$	${}^{0''}R_A = Rot(X, \psi)$

The rotation matrix ${}^0R_A$ representing roll (ψ) pitch (θ) yaw (ϕ) angles is

$$ {}^0R_A = {}^0R_{0'}\,{}^{0'}R_{0''}\,{}^{0''}R_A = \begin{bmatrix} C_\phi C_\theta & C_\phi S_\theta S_\psi - S_\phi C_\psi & C_\phi S_\theta C_\psi + S_\theta S_\psi \\ S_\phi C_\theta & S_\phi S_\theta S_\psi + C_\phi C_\psi & S_\phi S_\theta C_\psi - C_\phi S_\psi \\ -S_\theta & C_\theta S_\psi & C_\theta C_\psi \end{bmatrix} \tag{1.10} $$

1.5 Homogeneous Transformation Matrix

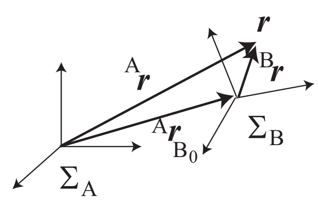


Fig. 1.5 Translation and rotation

In the kinematics of robotic system, the homogeneous transformation matrix which represents translational and rotational transformation between two coordinate frames is often used. The translational and rotational transformation of a vector $\boldsymbol{r}$ between two coordinate frames Σ_A and Σ_B is described by

$$ {}^A\boldsymbol{r} = {}^A\boldsymbol{r}_{B0} + {}^AR_B\,{}^B\boldsymbol{r} \tag{1.11} $$

where ${}^A\boldsymbol{r}_{B0}$ is the origin point vector of Σ_B in the Σ_A. We here introduce the notation of

$$ {}^A\boldsymbol{P} \triangleq \begin{bmatrix} {}^A\boldsymbol{r} \\ 1 \end{bmatrix}, \quad {}^B\boldsymbol{P} \triangleq \begin{bmatrix} {}^B\boldsymbol{r} \\ 1 \end{bmatrix}, \quad {}^AT_B \triangleq \begin{bmatrix} {}^AR_B & {}^A\boldsymbol{r}_{B0} \\ 0\,0\,0 & 1 \end{bmatrix} $$

then we can simply describe the transformation (1.11) by

$$ {}^A\boldsymbol{P} = {}^AT_B\,{}^B\boldsymbol{P} \tag{1.12} $$

The ${}^AT_B$ is called **"homogeneous transformation matrix"**.

[**Characteristics of homogeneous transformation matrix**]

$$ {}^AT_C = {}^AT_B\,{}^BT_C \tag{1.13} $$

$$ ({}^AT_B)^{-1} = {}^BT_A = \begin{bmatrix} ({}^AR_B)^T & -({}^AR_B)^T\,{}^A\boldsymbol{r}_{B0} \\ 0\,0\,0 & 1 \end{bmatrix} \tag{1.14} $$

For the later convenience, we also define the following specific homogeneous transformation matrices;

$$ T_{rot}(x, \theta) \triangleq \begin{bmatrix} & & & 0 \\ & Rot(x,\theta) & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.15} $$

$$T_{tran}(a,b,c) \triangleq \begin{bmatrix} & E_3 & & \begin{matrix} a \\ b \\ c \end{matrix} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1.16}$$

where E_3 is 3×3 unit matrix. Note that we can decompose ${}^{A}T_{B} = T_{tran}T_{rot}$ with only this order.

1.6 Modified Denavit Hartenberg Notation

To represent a position and an orientation of any part of a robot manipulator, we should set coordinate frames for each link of the robot properly. There are many ways to set the coordinate frames. One of popular way to set the coordinate frames is **"Modified Denavit Hartenberg Method"**. This subsection explains how to set the coordinate frames using four parameters for each link by the method.

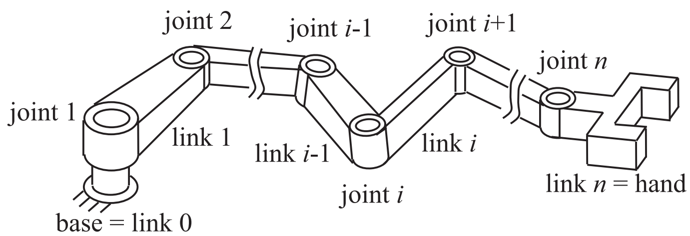


Fig. 1.6 Link coordinate frames

In this subsection, the following notation is used to distinguish various types of vectors.

$$\vec{\boldsymbol{a}} = \vec{\boldsymbol{b}} \quad : \quad \text{vector } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are identical } (\vec{\boldsymbol{a}} \parallel \vec{\boldsymbol{b}} \text{ and } |\vec{\boldsymbol{a}}| = |\vec{\boldsymbol{b}}|)$$

$$\vec{\boldsymbol{a}} \parallel \vec{\boldsymbol{b}} \quad : \quad \text{vector } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are parallel}$$

$$\vec{\boldsymbol{a}} \equiv \vec{\boldsymbol{b}} \quad : \quad \text{vector } \boldsymbol{a} \text{ and } \boldsymbol{b} \text{ are identical including the starting point}$$

1.6.1 Procedure for setting link coordinate frames

In this subsection, X_i, Y_i, Z_i mean axes of Σ_i coordinate frame. Vector $\vec{x}_i, \vec{y}_i, \vec{z}_i$ are unit vectors lying on the X_i, Y_i, Z_i axis each. The starting point of the vectors is the origin of Σ_i.

Step 1 Define the base as link 0. Then assign number for each link from the base. (link n = end link = hand)

Step 2 Assign number (1 to n) for each joint from the base.

Step 3 The axis of Z_i is the axis of joint i (rotational axis or translational axis). Define the direction of Z_i on the axis of joint i. Positive direction of the rotational axis (Z_i axis) should follow the right hand rule. Positive direction of the translational axis (Z_i axis) is the positive direction of the translational joint.

Step 4 Define X_{i-1} axis by the common perpendicular line of Z_{i-1} and Z_i. Set the origin of Σ_{i-1} as the intersection point of X_{i-1} and Z_{i-1}. Where the positive direction of X_{i-1} is defined as the cross product of two vectors as $\vec{x}_{i-1} \parallel (\vec{z}_{i-1} \times \vec{z}_i)$.

Step 5 Y_{i-1} axis is defined by the "right-handed system" rule.

Step 6 Set $\vec{z}_0 \equiv \vec{z}_1$ axis. The x_0 axis is arbitrary. In most cases, $\vec{x}_0 \equiv \vec{x}_1$ axis is recommended.

Step 7 X_n is arbitrary. In most cases, $\vec{x}_n = \vec{x}_{n-1}$ axis is recommended.

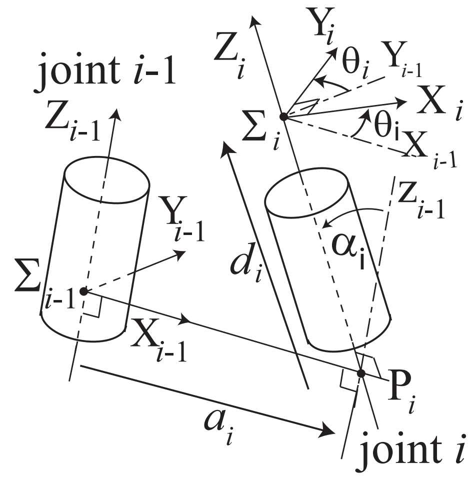


Fig. 1.7 Geometrical relation between Σ_{i-1} and Σ_i

1.6.2 Denavit Hartenberg parameters

Using each coordinate system on link i and the point P_i which is the intersection point of the common perpendicular (X_{i-1}) and Z_i axis (see Fig.(1.7)), we can find the following Denavit Hartenberg (D-H) parameters;

Step 8 Find P_i: the foot of X_{i-1} onto Z_i.

Step 9 Find a_i : length from Σ_{i-1} to P_i on X_{i-1} (positive or negative follows the direction of $\vec{x}_{i-1}$)

Step 10 Find α_i : angle from Z_{i-1} to Z_i around X_{i-1} (positive direction of the rotation axis is $\vec{x}_{i-1}$)

Step 11 Find d_i : distance from P_i to Σ_i (positive direction is $\vec{z}_i$) This is identical with joint variable q_i when the joint is the translational one. Note that d_i may include some offset value for such case (see 1.6.4).

Step 12 Find θ_i : angle from X_{i-1} to X_i around Z_i (positive direction of the rotation axis is $\vec{z}_i$) This is identical with joint variable q_i when the joint is the rotational one. Note that θ_i may include some offset value for such case (see 1.6.3).

Then the transformation from the previous coordinate frame Σ_{i-1} to the coordinate frame Σ_i is constructed by

1. translate a_i along $X_{i-1} : T_{tran}(a_i, 0, 0)$
2. rotate α_i around $X_{i-1} : T_{rot}(x_{i-1}, \alpha_i)$
3. translate d_i from P_i to $\Sigma_i : T_{tran}(0, 0, d_i)$
4. rotate θ_i around Z_{i-1} $(= Z_i) : T_{rot}(z_{i-1}, \theta_i)$

The total homogeneous transformation matrix from Σ_i to Σ_{i-1} is, then described by

$$
\begin{aligned}
{}^{i-1}T_i &= T_{tran}(a_i, 0, 0) T_{rot}(x_{i-1}, \alpha_i) T_{tran}(0, 0, d_i) T_{rot}(z_{i-1}, \theta_i) \\
&= \begin{bmatrix} C_{\theta_i} & -S_{\theta_i} & 0 & a_i \\ C_{\alpha_i} S_{\theta_i} & C_{\alpha_i} C_{\theta_i} & -S_{\alpha_i} & -d_i S_{\alpha_i} \\ S_{\alpha_i} S_{\theta_i} & S_{\alpha_i} C_{\theta_i} & C_{\alpha_i} & d_i C_{\alpha_i} \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (1.17)
\end{aligned}
$$

a_i, α_i, d_i, θ_i are called Denavit Hartenberg parameters.

1.6.3 Denavit Hartenberg parameters for rotational joint

When the joint-i is a revolution one, the D-H parameter θ_i contains joint variable q_i. When the q_i is 0 (initial state of the robot arm), there may be $\theta_i = \bar{\theta}$ as a "offset angle". So we should represent

$$\theta_i = \bar{\theta}_i + q_i$$

for the general case of rotational joint (see Fig.1.8).

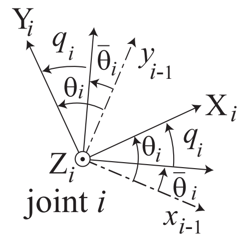


Fig. 1.8 Offset angle $\bar{\theta}_i$

1.6.4 Denavit Hartenberg parameters for prismatic joint

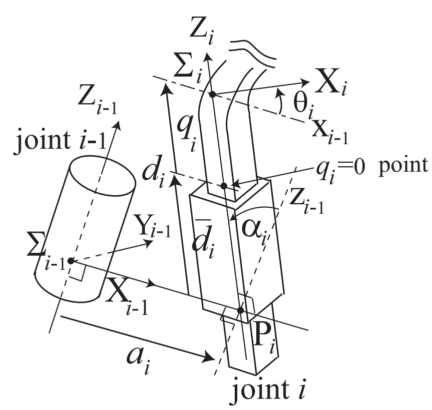


Fig. 1.9 Geometrical relation between Σ_{i-1} and Σ_i for prismatic joint

The definition of Denavit Hartenberg parameters for prismatic joints is as same as the one for the revolution joint. The homogeneous transformation matrix ${}^{i-1}T_i$ is also same. For the prismatic joint case, the parameter d_i contains joint variable q_i. When the q_i is 0 (initial state of the robot arm), there may be $d_i = \bar{d}$ as a "offset length". So we should represent

$$d_i = \bar{d}_i + q_i$$

for the general case of prismatic joint (see Fig.1.9).

1.7 Position and Orientation of Hand

The homogeneous transformation representing the relation between hand coordinate frame $\Sigma_h (= \Sigma_n)$ and base coordinate frame Σ_0 can be described by

$$ {}^0T_n = {}^0T_1\, {}^1T_2 \cdots {}^{n-1}T_h = \begin{bmatrix} {}^0R_h & {}^0\boldsymbol{r}_{h0} \\ 0\ 0\ 0 & 1 \end{bmatrix} \tag{1.18}$$

where the rotation matrix ${}^0R_h$ represents the orientation of the hand and ${}^0\boldsymbol{r}_{h0}$ represents origin point of the hand coordinate system in the reference of Σ_0 coordinate system.

1.8 Representation of Arbitrary Point of a Link

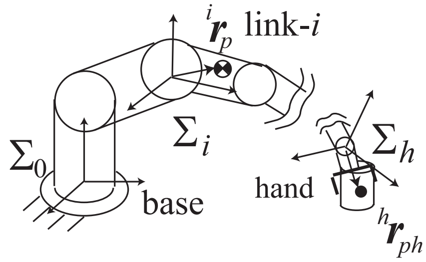


Fig. 1.10 Arbitrary point in link-i

Arbitrary point $\boldsymbol{r}_p$ in link i in the reference of base coordinate frame Σ_0 can be represented by homogeneous transformation using the link coordinate system as;

$$\begin{bmatrix} {}^0\boldsymbol{r}_p \\ 1 \end{bmatrix} = {}^0\boldsymbol{P}_p = {}^0 T_i \, {}^i\boldsymbol{P}_p = \begin{bmatrix} {}^0R_i & {}^0\boldsymbol{r}_{i0} \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} {}^i\boldsymbol{r}_p \\ 1 \end{bmatrix} \tag{1.19}$$

This equation is one of the general form of kinematics. For example, the point of sized object by hand ${}^0\boldsymbol{r}_{ph}$ in the base coordinate system can be represented by

$$\begin{bmatrix} {}^0\boldsymbol{r}_{ph} \\ 1 \end{bmatrix} = {}^0\boldsymbol{P}_{ph} = {}^0T_h(\boldsymbol{q})\, {}^i\boldsymbol{P}_{ph} = {}^0T_h(\boldsymbol{q}) \begin{bmatrix} {}^h\boldsymbol{r}_{ph} \\ 1 \end{bmatrix}$$

which means the point of sized object by hand ${}^0\boldsymbol{r}_{ph}$ is represented by joint variables $\boldsymbol{q}$ and constant vector ${}^h\boldsymbol{r}_{ph}$.

1.9 Numerical Method for Inverse Kinematics Calculation

From Eq(1.19), we see that forward kinematics equation can be represented by the form of

$$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q}) \tag{1.20}$$

where $\boldsymbol{r} \in \Re^n$ is position (and orientation) of end-effector and $\boldsymbol{q} \in \Re^n$ is joint variable (included in θ_i or d_i in D-H parameters). By differentiating both sides of the equation, we can write

$$d\boldsymbol{r} = \frac{\partial \boldsymbol{f}(\boldsymbol{q})}{\partial \boldsymbol{q}} d\boldsymbol{q} = J(\boldsymbol{q}) d\boldsymbol{q} \tag{1.21}$$

Then we have the following difference equation which represents inverse kinematics.

$$d\boldsymbol{q} = J^{-1}(\boldsymbol{q}) d\boldsymbol{r} \tag{1.22}$$

[**An algorithm for calculating inverse kinematics solution** ($\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{r})$)]

step 1) Give the value $\boldsymbol{q}_0$ which is an approximate value of actual $\boldsymbol{q}$. Calculate $\boldsymbol{r}_0 = \boldsymbol{f}(\boldsymbol{q}_0)$

step 2) $i = 1$

step 3) Calculate $\boldsymbol{q}_i = \boldsymbol{q}_{i-1} + kJ^{-1}(\boldsymbol{q}_{i-1})(\boldsymbol{r} - \boldsymbol{r}_{i-1})$
where k is positive small value.

step 4) Calculate $\boldsymbol{r}_i = \boldsymbol{f}(\boldsymbol{q}_i)$: if $\boldsymbol{r} \approx \boldsymbol{r}_i$, then stop the calculation.

step 5) $i = i + 1$, goto step 3).

where

$$J(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{q}} = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \cdots & \frac{\partial f_1}{\partial q_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial q_1} & \cdots & \frac{\partial f_n}{\partial q_n} \end{bmatrix} \tag{1.23}$$

1.10 Inverse Kinematics Calculation for 2-Link Arm

The numerical calculation method in the previous subsection has disadvantages. For example, bad initial approximation $\boldsymbol{q}_0$ may lead no convergence to real value. Thus the analytical form of the inverse kinematics solution is desirable. However, getting the analytical solution for the general case of robot manipulator is impossible, because of the non-linear equation of the forward kinematics.

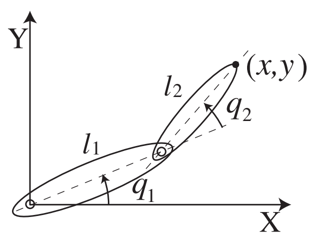


Fig. 1.11 Two-link plane manipulator

Although the fact, there are some analytical solutions for some specific robot arms. For plane type 2-link manipulator (as shown in Fig.(1.11)), we can calculate the joint variable (q_1, q_2) from (x, y) directly:

$$\begin{cases} q_1 = \text{atan2}(y, x) \mp \text{atan2}(k,\ l_1^2 + x^2 + y^2 - l_2^2) \\ q_2 = \pm\text{atan2}(k,\ -(l_1^2 + l_2^2 - x^2 - y^2)) \end{cases} \tag{1.24}$$

where $k = \sqrt{(x^2 + y^2 + l_1^2 + l_2^2)^2 - 2((x^2 + y^2)^2 + l_1^4 + l_2^4)}$.
This result is a basic for the analytical solutions for specific robot arms.

1.11 Differential Representation of Orientation

There are two types of representation for "velocity of orientation angles";

(I) The use of differential for Euler angles $= \dot{\boldsymbol{\eta}}$

(II) The use of angular velocity $= \boldsymbol{\omega}$

Note that the Euler angles $(\boldsymbol{\eta} = (\phi, \theta, \psi))$ are not vector, so the velocities of them $\dot{\boldsymbol{\eta}}$ are not vector.

1.12 Definition of Angular Velocity

In the definition of angular velocity, a rigid body is assumed to be rotating in three dimensional space. In addition, a point $\boldsymbol{p}$ is on the rigid body. Then the angular velocity is defined as followings.
1) The angular velocity $\boldsymbol{\omega}$ is "a vector", thus it has the elements for X,Y,Z axis. The vector is uniquely defined by its direction and its magnitude.
2) The direction of $\boldsymbol{\omega}$ is the direction of the rotating axis of the rigid body and the point $\boldsymbol{p}$ (see Fig.1.12).
3) The magnitude is the speed of the rotation $\dot{\theta}$ ($|\boldsymbol{\omega}| = \dot{\theta}$).

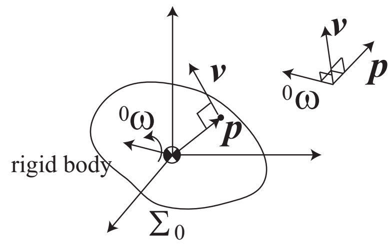


Fig. 1.12 Definition of angular velocity $^0\omega$

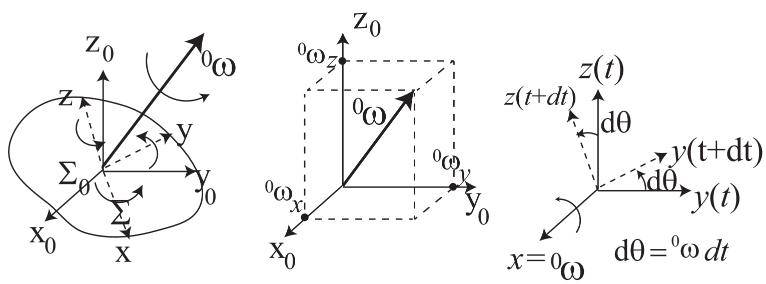


Fig. 1.13 Angular velocity ω

Then angular velocity $\boldsymbol{\omega}$ can be written as

$$\boldsymbol{\omega} = \begin{bmatrix} \frac{\omega_x}{|\omega|} \\ \frac{\omega_y}{|\omega|} \\ \frac{\omega_z}{|\omega|} \end{bmatrix} = \boldsymbol{i}_\omega \dot{\theta} \tag{1.25}$$

where $\boldsymbol{i}_\omega$ is the unit vector along $\boldsymbol{\omega}$.

When a point $\boldsymbol{p}$ is rotating around $\boldsymbol{\omega}$ with not changing its magnitude and its velocity is $\boldsymbol{v}$ ($\dot{\boldsymbol{p}}=\boldsymbol{v}$), then

$$\boldsymbol{\omega} = \frac{\boldsymbol{p}}{|\boldsymbol{p}|} \times \frac{\boldsymbol{v}}{|\boldsymbol{p}|} \tag{1.26}$$

Or equivalently the velocity $\boldsymbol{v}$ can be written as

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{p} \tag{1.27}$$

These relations are easily proved by the definition.

Especially, when Σ coordinate frame is fixed with the rigid object as in Fig.1.13 left, each axis x, y, z of Σ rotates around $^0\boldsymbol{\omega}$. The angular velocity $^0\boldsymbol{\omega}$ is a vector, thus the vector can be decomposed into each element ($^0\omega_x$, $^0\omega_y$, $^0\omega_z$) as in the middle of Fig.1.13.

As a special case, when the $^0\boldsymbol{\omega}$ axis is same with x axis as in Fig.1.13 right, the y axis at time t $y(t)$ and z axis at time t, $z(t)$ rotate around $x =^0 \boldsymbol{\omega}$ wth $d\theta$, then $d\theta$ is calculated by

$$d\theta = |^0\boldsymbol{\omega}|dt = {}^0\omega_x dt \tag{1.28}$$

or

$$\dot{\theta} = \frac{d\theta}{dt} = {}^0\omega_x \tag{1.29}$$

because $^0\boldsymbol{\omega}$ has only ω_x in this case. This is also the difinition of magnitude for $^0\boldsymbol{\omega}$. Note that the integral of $^0\boldsymbol{\omega}$ has no physical meaning.

1.13 Relationship Between Euler Angles and Angular Velocity

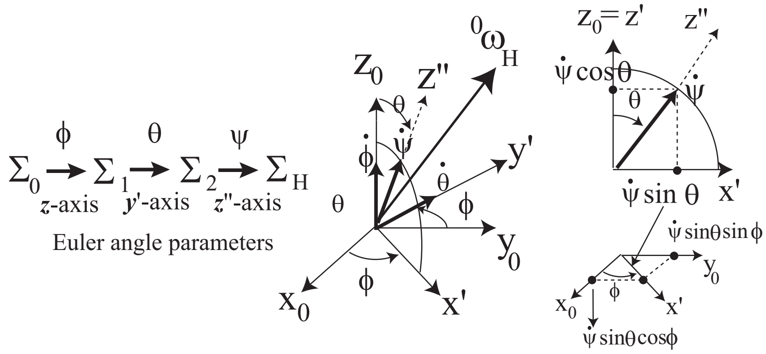


Fig. 1.14 Relationship of the velocity for Euler angles and angular velocity

The relationship of the velocities of Euler angle parameters and angular velocity is obtained by the followings. The angular velocity ${}^0\boldsymbol{\omega}_H$ for the Euler parameters is obtained by the sum of each angular velcity at each step as,

$$ {}^0\boldsymbol{\omega}_H = {}^0\boldsymbol{\omega}_{0\to1} + {}^0\boldsymbol{\omega}_{1\to2} + {}^0\boldsymbol{\omega}_{2\to H} \tag{1.30} $$

By the definition of Euler angles, initial coordinate frame Σ_0 is rotated around Z_0 axis with ϕ at speed $\dot{\phi}$, the angular velocity ${}^0\boldsymbol{\omega}_{0\to1}$ for the rotation is

$$ {}^0\boldsymbol{\omega}_{0\to1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\phi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} \tag{1.31} $$

Similarly, ${}^0\boldsymbol{\omega}_{1\to2}$ and ${}^0\boldsymbol{\omega}_{2\to H}$ are calculated using $\dot{\theta}$ and $\dot{\psi}$ as

$$ {}^0\boldsymbol{\omega}_{1\to2} = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix} \dot{\theta} = \begin{bmatrix} -\sin\phi\dot{\theta} \\ \cos\phi\dot{\theta} \\ 0 \end{bmatrix} \tag{1.32} $$

$$ {}^0\boldsymbol{\omega}_{2\to H} = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix} \dot{\psi} = \begin{bmatrix} \sin\theta\cos\phi\dot{\psi} \\ \sin\theta\sin\phi\dot{\psi} \\ \cos\theta\dot{\psi} \end{bmatrix} \tag{1.33} $$

Totally, thus, ${}^0\boldsymbol{\omega}_H$ is described using the Eular parameters and their velocities as

$$ {}^0\boldsymbol{\omega}_H = \begin{bmatrix} 0 & -S_\phi & S_\theta C_\phi \\ 0 & C_\phi & S_\theta S_\phi \\ 1 & 0 & C_\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \Omega(\phi,\theta)\,{}^0\dot{\boldsymbol{\eta}}_H \tag{1.34} $$

If matrix Ω is regular,

$$ {}^0\dot{\boldsymbol{\eta}}_H = \Omega^{-1}(\phi,\theta)\,{}^0\boldsymbol{\omega}_H \tag{1.35} $$

1.14 Differential Relation of Position and Orientation

$$ \boldsymbol{r} = \begin{bmatrix} \boldsymbol{p}_H \\ \boldsymbol{\eta}_H \end{bmatrix} \quad \left\{ \begin{array}{lll} \boldsymbol{p}_H : & \text{position vector of hand} & = \boldsymbol{f}_1(\boldsymbol{q}) \\ \boldsymbol{\eta}_H : & \text{orientation of hand (Euler parameter)} & = \boldsymbol{f}_2(\boldsymbol{q}) \end{array} \right. $$

By differentiating $\boldsymbol{r}$ formally , we have

$$\dot{\boldsymbol{r}} = \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \dot{\boldsymbol{\eta}}_H \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{f}_1}{\partial \boldsymbol{q}} \\ \frac{\partial \boldsymbol{f}_2}{\partial \boldsymbol{q}} \end{bmatrix} \dot{\boldsymbol{q}} = J(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{1.36}$$

where H means hand. On the other hand, by setting

$$\dot{\boldsymbol{r}}_\omega = \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \boldsymbol{\omega}_H \end{bmatrix}$$

we have

$$\dot{\boldsymbol{r}}_\omega = \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \Omega\dot{\boldsymbol{\eta}}_H \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & \Omega \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \dot{\boldsymbol{\eta}}_H \end{bmatrix} = \begin{bmatrix} I_3 & 0 \\ 0 & \Omega \end{bmatrix} J(\boldsymbol{q})\dot{\boldsymbol{q}} = J_\omega(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{1.37}$$

Thus we have two types of Jacobian $J(\boldsymbol{q})$ and $J_\omega(\boldsymbol{q})$.

1.15 Summary of Kinematics

	forward kinematics	inverse kinematics
position/angle	$\boldsymbol{r} = \boldsymbol{f}(\boldsymbol{q})$	$\boldsymbol{q} = \boldsymbol{f}^{-1}(\boldsymbol{r})$
velocity	$\dot{\boldsymbol{r}} = \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \dot{\boldsymbol{\eta}}_H \end{bmatrix} = J\dot{\boldsymbol{q}}$ or $\dot{\boldsymbol{r}}_\omega = \begin{bmatrix} \dot{\boldsymbol{p}}_H \\ \dot{\boldsymbol{\omega}}_H \end{bmatrix} = J_\omega\dot{\boldsymbol{q}}$	$\dot{\boldsymbol{q}} = J^{-1}\dot{\boldsymbol{r}}$ or $\dot{\boldsymbol{q}} = J_\omega^{-1}\dot{\boldsymbol{r}}_\omega$
acceleration	$\ddot{\boldsymbol{r}} = J\ddot{\boldsymbol{q}} + \dot{J}\dot{\boldsymbol{q}}$ or $\ddot{\boldsymbol{r}}_\omega = J_\omega\ddot{\boldsymbol{q}} + \dot{J}_\omega\dot{\boldsymbol{q}}$	$\ddot{\boldsymbol{q}} = J^{-1}(\ddot{\boldsymbol{r}} - \dot{J}J^{-1}\dot{\boldsymbol{r}})$ or $\ddot{\boldsymbol{q}}_\omega = J_\omega^{-1}(\ddot{\boldsymbol{r}}_\omega - \dot{J}_\omega J_\omega^{-1}\dot{\boldsymbol{r}_\omega})$

Chapter 2

STATICS

Using the kinematic relation of joint variable $\boldsymbol{q}$ and workspace variable $\boldsymbol{r}$ and principle of virtual work, we can discuss the relation of joint torques (or joint forces) and adding force and moment at hand part. This is called "statics".

2.1 Principle of Virtual Work

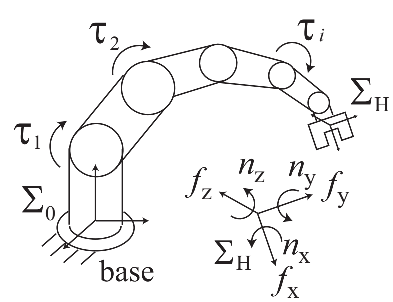


Fig. 2.1 Force and moment in hand coordinate frame

We use the following notations.

$$\boldsymbol{m} = \begin{bmatrix} f_x \\ f_y \\ f_z \\ n_x \\ n_y \\ nz \end{bmatrix} = \begin{bmatrix} {}^0\boldsymbol{f}_H \\ {}^0\boldsymbol{n}_H \end{bmatrix}, \quad \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_n \end{bmatrix}, \ \tau_i : \text{joint torque}$$

where ${}^0\boldsymbol{f}_H$, ${}^0\boldsymbol{n}_H$ are adding force and moment to hand.
By principle of virtual work (the total work by the virtual displacement is zero), we have

$$d\boldsymbol{q}^T\,\boldsymbol{\tau} - (d\boldsymbol{r}_w)^T\,\boldsymbol{m} = 0 \tag{2.1}$$

Using $\dot{\boldsymbol{r}}_w = J_w\dot{\boldsymbol{q}} \rightarrow d\boldsymbol{r}_w = J_w d\boldsymbol{q}$,

$$\boldsymbol{\tau} = J_w^T\,\boldsymbol{m} \tag{2.2}$$

Note that we use J_w in the statistics equation. Generally we have the following relations between Cartesian coordinates and joint coordinates,

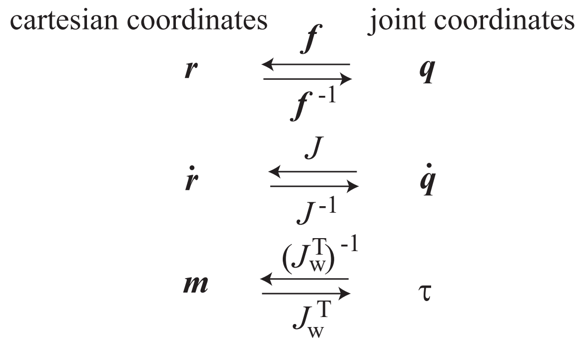


Fig. 2.2 Cartesian coordinates and joint coordinates

2.2 Transformation of Force and Moment

We denote the force and moment ${}^H\boldsymbol{m}_H$ in hand coordinate frame as

$$ {}^H\boldsymbol{m}_H = \begin{bmatrix} {}^H\boldsymbol{f}_H \\ {}^H\boldsymbol{n}_H \end{bmatrix} $$

Using the rotation matrix ${}^0R_H$ and position vector of hand-origin ${}^0\boldsymbol{p}_H$, we can describe the force and moment in base coordinate frame as

$$ {}^0\boldsymbol{f}_H = {}^0R_H \, {}^H\boldsymbol{f}_H \tag{2.3} $$

$$ {}^0\boldsymbol{n}_H = {}^0R_H \, {}^H\boldsymbol{n}_H + {}^0\boldsymbol{p}_H \times {}^0\boldsymbol{f}_H \tag{2.4} $$

This equation is rewritten by the form of matrix·vector as

$$ {}^0\boldsymbol{n}_H =^0 R_H \, {}^H\boldsymbol{n}_H + [{}^0\boldsymbol{p}_H\times]^0R_H \, {}^H\boldsymbol{f}_H \tag{2.5} $$

where

$$ \boldsymbol{a} \times \boldsymbol{b} = [\boldsymbol{a}\times]\boldsymbol{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} $$

Then we have the following transformation formula of force and moment between hand coordinate frame and base coordinate frame.

$$ {}^0\boldsymbol{m}_H = \begin{bmatrix} {}^0\boldsymbol{f}_H \\ {}^0\boldsymbol{n}_H \end{bmatrix} = \begin{bmatrix} {}^0R_H & 0 \\ [{}^0\boldsymbol{p}_H] \times {}^0R_H & {}^0R_H \end{bmatrix} \begin{bmatrix} {}^H\boldsymbol{f}_H \\ {}^H\boldsymbol{n}_H \end{bmatrix} = {}^0\Gamma_H \, {}^H\boldsymbol{m}_H \tag{2.6} $$

where ${}^0\Gamma_H$ is the transformation matrix of force and moment. Using the result of previous section,

$$ \boldsymbol{\tau} = J_w^T \, {}^0\Gamma_H \, {}^H\boldsymbol{m}_H \tag{2.7} $$

Chapter 3

DYNAMICS BY LAGRANGE EQUATION

3.1 Lagrange Equation

Using the definition of Lagrangian $\mathcal{L} = K - P$ (see Appendix G),

$$\tau_i = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right] - \frac{\partial \mathcal{L}}{\partial q_i} \tag{3.1}$$

where q_i is generalized coordinates and τ_i is generalized force. This is called Lagrange equation or Euler-Lagrange equation of motion. Using the notation that K is kinematics energy and P is potential energy, the Lagrange equation is

$$\tau_i = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{q}_i}\right] - \frac{\partial K}{\partial q_i} + \frac{\partial P}{\partial q_i} \tag{3.2}$$

3.2 Kinetic Energy

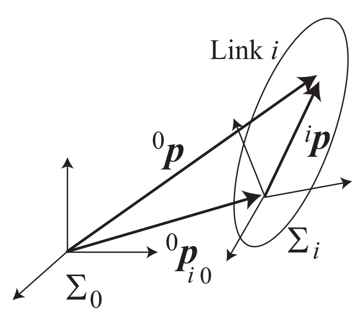


Fig. 3.1 Kinetic energy of link i

Representing kinetic energy of link i by K_i, the total kinetic energy of manipulator can be described by

$$K = \sum_{i=1}^{n} K_i \tag{3.3}$$

The kinetic energy of small part dK_i corresponding to the small mass dm_i is

$$\begin{aligned} dK_i &= \frac{1}{2}({}^0\dot{\boldsymbol{P}})^T\,({}^0\dot{\boldsymbol{P}})dm && (3.4)\\ &= \frac{1}{2}\mathrm{tr}\left[({}^0\dot{\boldsymbol{P}})\,({}^0\dot{\boldsymbol{P}})^T\right]dm && (3.5)\end{aligned}$$

where ${}^0\boldsymbol{P} = [{}^i p_x, {}^i p_y, {}^i p_z, 1]^T$, thus ${}^0\dot{\boldsymbol{P}} = [{}^i\dot{p}_x, {}^i\dot{p}_y, {}^i\dot{p}_z, 0]^T$. Using the relation,

$$
\begin{aligned}
{}^0\dot{\boldsymbol{P}} &= \frac{d}{dt}({}^0T_i\, {}^i\boldsymbol{P}) = {}^0\dot{T}_i\, {}^i\boldsymbol{P} + {}^0T_i\, {}^i\dot{\boldsymbol{P}} && (3.6)\\
&= {}^0\dot{T}_i\, {}^i\boldsymbol{P} && (3.7)
\end{aligned}
$$

we have the following kinetic energy for small part dm

$$
dK_i = \frac{1}{2}\text{tr}\left[({}^0\dot{T}_i)\, ({}^i\boldsymbol{P})\, ({}^i\boldsymbol{P})^T\, {}^0\dot{T}_i\right] dm
$$

$$
K_i = \int_{\text{Link}-i} dK_i = \frac{1}{2}\text{tr}\left[({}^0\dot{T}_i)\, \int_{\text{Link}-i} ({}^i\boldsymbol{P})\, ({}^i\boldsymbol{P})^T dm\, ({}^0\dot{T}_i)^T\right] \tag{3.8}
$$

where

$$
\int_{\text{Link}-i} ({}^i\boldsymbol{P})\, ({}^i\boldsymbol{P})^T dm = {}^i H_i
$$

$$
{}^iH_i = \begin{bmatrix}
\int_{\text{Link}-i} {}^ip_x^2 dm & \int_{\text{Link}-i} {}^ip_x\, {}^ip_y dm & \int_{\text{Link}-i} {}^ip_x\, {}^ip_z dm & \int_{\text{Link}-i} {}^ip_x dm \\
\int_{\text{Link}-i} {}^ip_y\, {}^ip_x dm & \int_{\text{Link}-i} {}^ip_y^2 dm & \int_{\text{Link}-i} {}^ip_y\, {}^ip_z dm & \int_{\text{Link}-i} {}^ip_y dm \\
\int_{\text{Link}-i} {}^ip_z\, {}^ip_x dm & \int_{\text{Link}-i} {}^ip_z\, {}^ip_y dm & \int_{\text{Link}-i} {}^ip_z^2 dm & \int_{\text{Link}-i} {}^ip_z dm \\
\int_{\text{Link}-i} {}^ip_x dm & \int_{\text{Link}-i} {}^ip_y dm & \int_{\text{Link}-i} {}^ip_z dm & \int_{\text{Link}-i} dm
\end{bmatrix} \tag{3.9}
$$

3.3 Pseudo Inertia Matrix

Using the inertia moment around x-axis,

$$
I_{ixx} = \int_{\text{Link}-i} ({}^ip_y^2 + {}^ip_z^2) dm \tag{3.10}
$$

The elements of ${}^iH_i$ can be represented by similar notations for the inertia moment.

$$
\begin{aligned}
\int_{\text{Link}-i} {}^ip_x^2 dm &= \frac{1}{2}(I_{iyy} + I_{izz} - I_{ixx}) && (3.11)\\
H_{ixy} = H_{iyx} &= \int_{\text{Link}-i} {}^ip_x\, {}^ip_y dm && (3.12)\\
m_i &= \int_{\text{Link}-i} dm && (3.13)\\
{}^is_{ix} &= \frac{1}{m_i}\int_{\text{Link}-i} {}^ip_x dm && (3.14)
\end{aligned}
$$

Then we can represent the ${}^iH_i$ as

$$
{}^iH_i = H_i = \begin{bmatrix}
\frac{1}{2}(I_{iyy} + I_{izz} - I_{ixx}) & H_{ixy} & H_{ixz} & m_i {}^is_{ix} \\
H_{ixy} & \frac{1}{2}(I_{ixx} + I_{izz} - I_{iyy}) & H_{iyz} & m_i {}^is_{iy} \\
H_{ixz} & H_{iyz} & \frac{1}{2}(I_{ixx} + I_{iyy} - I_{izz}) & m_i {}^is_{iz} \\
m_i {}^is_{ix} & m_i {}^is_{iy} & m_i {}^is_{iz} & m_i
\end{bmatrix} \tag{3.15}
$$

As a result, the kinetic energy of link i is

$$
K = \sum_{i=1}^{n} K_i = \frac{1}{2}\sum_{i=1}^{n} \text{tr}({}^0\dot{T}_i H_i ({}^0\dot{T}_i)^T) \tag{3.16}
$$

For your information: Inertia tensor is defined by $M = I\boldsymbol{\omega}$ as

$$I = \begin{bmatrix} I_{xx} & -H_{xy} & -H_{xz} \\ -H_{xy} & I_{yy} & -H_{yz} \\ -H_{xz} & -H_{yz} & I_{zz} \end{bmatrix} \tag{3.17}$$

where M is angular momentum and $\boldsymbol{\omega}$ is angular velocity.

3.3.1 Calculation of $\frac{d}{dt}\left[\frac{\partial K}{\partial \dot{q}_i}\right]$

From (3.16),

$$\frac{\partial K}{\partial \dot{q}_i} = \frac{1}{2}\sum_{k=1}^{n} \text{tr}\frac{\partial}{\partial \dot{q}_i}\left[{}^0\dot{T}_k H_k ({}^0\dot{T}_k)^T\right]$$

Note that subscript i is changed to k. Then, we have

$$\frac{d}{dt}\left[\frac{\partial K}{\partial \dot{q}_i}\right] = \sum_{k=1}^{n} \text{tr}\left\{\frac{d}{dt}\left[\frac{\partial {}^0\dot{T}_k}{\partial \dot{q}_i}\right] H_k ({}^0\dot{T}_k)^T + \frac{\partial {}^0\dot{T}_k}{\partial \dot{q}_i} H_k ({}^0\ddot{T}_k)^T\right\} \tag{3.18}$$

In the above derivation, we use the following formulae

$$\begin{cases} (ABC)^T = C^T B^T A^T \\ \text{tr}(A) = \text{tr}(A^T) \\ H_k \text{ is symmetric and constant on time } t \end{cases} \tag{3.19}$$

3.3.2 Some Preliminaries for Derivation

$$ {}^0\dot{T}_i = \frac{d}{dt}\left[{}^0T_i\right] = \sum_{l=1}^{i} \frac{\partial {}^0T_i}{\partial q_l}\dot{q}_l \tag{3.20}$$

$$\frac{\partial {}^0\dot{T}_i}{\partial \dot{q}_k} = \frac{\partial {}^0T_i}{\partial q_k} \tag{3.21}$$

$$\frac{d}{dt}\left[\frac{\partial {}^0\dot{T}_i}{\partial \dot{q}_k}\right] = \frac{d}{dt}\left[\frac{\partial {}^0T_i}{\partial q_k}\right] = \frac{\partial}{\partial q_k}\left(\sum_{l=1}^{i} \frac{\partial {}^0T_i}{\partial q_l}\dot{q}_l\right) = \frac{\partial {}^0\dot{T}_i}{\partial q_k} \tag{3.22}$$

Using the above equations,

$$\frac{d}{dt}\left[\frac{\partial K}{\partial \dot{q}_i}\right] = \sum_{k=i}^{n} \text{tr}\left(\frac{\partial {}^0\dot{T}_k}{\partial q_i} H_k ({}^0\dot{T}_k)^T + \frac{\partial {}^0T_k}{\partial q_i} H_k\, ({}^0\ddot{T}_k)^T\right) \tag{3.23}$$

where

$$ {}^0\ddot{T}_i = \frac{d}{dt}\sum_{l=1}^{i} \frac{\partial {}^0T_i}{\partial q_l}\dot{q}_l = \sum_{l=1}^{i}\sum_{m=1}^{i} \frac{\partial^2\, {}^0T_i}{\partial q_m \partial q_l}\dot{q}_l\,\dot{q}_m + \sum_{l=1}^{i} \frac{\partial {}^0T_i}{\partial q_l}\ddot{q}_l \tag{3.24}$$

3.3.3 Calculation of $\frac{\partial K}{\partial q_i}$

$$\begin{aligned} \frac{\partial K}{\partial q_i} &= \frac{1}{2}\sum_{k=1}^{n} \text{tr}\frac{\partial}{\partial q_i}\left[{}^0\dot{T}_k H_k ({}^0\dot{T}_k)^T\right] \\ &= \sum_{k=i}^{n} \text{tr}\left[\frac{\partial {}^0\dot{T}_k}{\partial q_i} H_k ({}^0\dot{T}_k)^T\right] \end{aligned} \tag{3.25}$$

3.3.4 Calculation of $\dfrac{\partial P}{\partial q_i}$

The definition of potential energy of link is

$$P = -\sum_{k=1}^{n} m_k ({}^0\boldsymbol{g})^T \left[{}^0T_k \, {}^k\boldsymbol{s}_k\right] \tag{3.26}$$

$$\frac{\partial P}{\partial q_i} = -\sum_{k=i}^{n} m_k ({}^0\boldsymbol{g})^T \left[\frac{\partial {}^0T_k}{\partial q_i} \, {}^k\boldsymbol{s}_k\right] \tag{3.27}$$

3.3.5 Calculation of τ_i

Using Eq.(3.2), Eq.(3.23), Eq.(3.24), Eq.(3.25) and Eq.(3.27), τ_i is calculated by

$$\tau_i = \sum_{k=i}^{n}\sum_{l=1}^{k} \mathrm{tr}\left[\frac{\partial {}^0T_k}{\partial q_i} H_k (\frac{\partial {}^0T_k}{\partial q_l})^T\right] \ddot{q}_l + \sum_{k=i}^{n}\sum_{l=1}^{k}\sum_{m=1}^{k} \mathrm{tr}\left[\frac{\partial {}^0T_k}{\partial q_i} H_k (\frac{\partial^2 \, {}^0T_k}{\partial q_l \partial q_m})^T\right] \dot{q}_l \dot{q}_m - \sum_{k=i}^{n} m_k ({}^0\boldsymbol{g})^T \left(\frac{\partial {}^0T_k}{\partial q_i} \, {}^k\boldsymbol{s}_k\right) \tag{3.28}$$

By setting

$$\begin{cases} M_{ij} &= \sum_{k=\max(i,j)}^{n} \mathrm{tr}\left[\frac{\partial {}^0T_k}{\partial q_i} H_k (\frac{\partial {}^0T_k}{\partial q_j})^T\right] \\ h_i &= \sum_{k=i}^{n}\sum_{l=1}^{k}\sum_{m=1}^{k} \mathrm{tr}\left[\frac{\partial {}^0T_k}{\partial q_i} H_k (\frac{\partial^2 \, {}^0T_k}{\partial q_l \partial q_m})^T\right] \dot{q}_l \dot{q}_m \\ g_i &= -\sum_{k=i}^{n} m_k ({}^0\boldsymbol{g})^T (\frac{\partial {}^0T_k}{\partial q_i} \, {}^k\boldsymbol{s}_k) \end{cases}$$

we can describe the dynamics equation by the form of

$$\boldsymbol{\tau} = M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \tag{3.29}$$

As a result, we have only to calculate M and $\boldsymbol{g}$ to get $\boldsymbol{\tau}$. For the calculation of M_{ij}, we calculate the following $\frac{\partial {}^0T_i}{\partial q_j}$ by

$$\frac{\partial {}^0T_i}{\partial q_j} = {}^0T_1 \, {}^1T_2 \, \cdots \, {}^{j-1}T_j \, Q_j {}^jT_{j+1} \, \cdots {}^{i-1}T_i \quad (j < i) \tag{3.30}$$

where

$$Q_j = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (for revolute joint)} \qquad Q_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ (for prismatic joint)}$$

3.3.6 Another Derivation of Dynamics Using the Inertia Moment Matrix and Lagrange Equation

The center of gravity point ${}^0\boldsymbol{s}_i$ for link -i is calculated by the forward kinematics

$${}^0\boldsymbol{s}_i = {}^0R_i {}^i\boldsymbol{s}_i = \boldsymbol{f}_{si}(q_1, q_2, \cdots, q_i) = \boldsymbol{f}_{si}(\boldsymbol{q}_i) \tag{3.31}$$

By taking the derivative

$${}^0\dot{\boldsymbol{s}}_i = \frac{\partial \boldsymbol{f}_{si}}{\partial \boldsymbol{q}_i} = J_{si}(\boldsymbol{q}_i)\dot{\boldsymbol{q}}_i \tag{3.32}$$

Similarly ${}^0\boldsymbol{\omega}_i$ is also described by

$${}^0\boldsymbol{\omega}_i = J_{\omega i}(\boldsymbol{q}_i)\dot{\boldsymbol{q}}_i \tag{3.33}$$

By denoting the kinetic energy K_i for limk$-i$ can be calculated with

$$K_i = \frac{1}{2}m_i {v_i}^2 + \frac{1}{2}I{\omega_i}^2 \tag{3.34}$$

More precisely, using (3.32) and (3.33),

$$\begin{aligned} K_i &= \frac{1}{2}m_i {}^0\dot{\boldsymbol{s}}_i^{\mathrm{T}}\ {}^0\dot{\boldsymbol{s}}_i + \frac{1}{2}{}^0\boldsymbol{\omega}_i^{\mathrm{T}}\ \hat{I}_i {}^0\boldsymbol{\omega}_i && (3.35) \\ &= \frac{1}{2}m_i \dot{\boldsymbol{q}}_i^{\mathrm{T}} J_{si}^{\mathrm{T}}\ J_{si}\dot{\boldsymbol{q}}_i + \frac{1}{2}\dot{\boldsymbol{q}}_i^{\mathrm{T}} J_{\omega i}^{\mathrm{T}}\ \hat{I}_i J_{\omega i}\dot{\boldsymbol{q}}_i && (3.36) \\ &= \frac{1}{2}\dot{\boldsymbol{q}}_i^{\mathrm{T}}(m_i J_{si}^{\mathrm{T}}\ J_{si} + J_{\omega i}^{\mathrm{T}}\ \hat{I}_i J_{\omega i})\dot{\boldsymbol{q}}_i && (3.37) \\ &= \frac{1}{2}\dot{\boldsymbol{q}}_i^{\mathrm{T}} M_i(\boldsymbol{q})\dot{\boldsymbol{q}}_i && (3.38) \end{aligned}$$

where $\hat{I}_i$ is the inertia tensor around the axis of center of gravity point in link$-i$ with reference to the Σ_0 coordinate frame, $M_i(\boldsymbol{q}) = m_i J_{si}^{\mathrm{T}}\ J_{si} + J_{\omega i}^{\mathrm{T}}\ \hat{I}_i J_{\omega i}$.
The potential energy P_i for link$-i$ is (when the Z_0 axis is $-g$ direction)

$$P_i = mgh_i = m_i \boldsymbol{g}^{\mathrm{T}}\ {}^0\boldsymbol{s}_i = m_i[0\ 0\ \ -g]\begin{bmatrix} 0 \\ 0 \\ h_i(\boldsymbol{q}_i) \end{bmatrix} \tag{3.39}$$

The the total kinetic energy K and the total potential energy P is

$$K = \sum K_i = \frac{1}{2}\dot{\boldsymbol{q}}_1^{\mathrm{T}} M_1(\boldsymbol{q})\dot{\boldsymbol{q}}_1 + \cdots \frac{1}{2}\dot{\boldsymbol{q}}_n^{\mathrm{T}} M_n(\boldsymbol{q})\dot{\boldsymbol{q}}_n = \frac{1}{2}\dot{\boldsymbol{q}}^{\mathrm{T}} M(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.40}$$

$$P = \sum P_i = m_1 \boldsymbol{g}^{\mathrm{T}}\ {}^0\boldsymbol{s}_1(\boldsymbol{q}_1) + \cdots m_n \boldsymbol{g}^{\mathrm{T}}\ {}^0\boldsymbol{s}_n(\boldsymbol{q}_n) \tag{3.41}$$

where $M = M_1 + \cdots M_n$ is called inertia moment matrix, and $\boldsymbol{q}_i = \boldsymbol{q}$.
Using the Lagrange function $\mathcal{L} = K - P$, joint torque $\boldsymbol{\tau}$ is

$$\boldsymbol{\tau} = \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial \mathcal{L}}{\partial \boldsymbol{q}} \tag{3.42}$$

Or

$$\boldsymbol{\tau} = \frac{d}{dt}\left[\frac{\partial K}{\partial \dot{\boldsymbol{q}}}\right] - \frac{\partial K}{\partial \boldsymbol{q}} + \frac{\partial P}{\partial \boldsymbol{q}} \tag{3.43}$$

Using (3.40) and (3.41)

$$\begin{aligned} \boldsymbol{\tau} &= \frac{d}{dt}[M(\boldsymbol{q})\dot{\boldsymbol{q}}] - \frac{\partial K}{\partial \boldsymbol{q}} + \frac{\partial P}{\partial \boldsymbol{q}} && (3.44) \\ &= \dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + M(\boldsymbol{q})\ddot{\boldsymbol{q}} - \frac{\partial K}{\partial \boldsymbol{q}} + \frac{\partial P}{\partial \boldsymbol{q}} && (3.45) \\ &= M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) && (3.46) \end{aligned}$$

where

$$\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \dot{M}\dot{\boldsymbol{q}} - \frac{\partial K}{\partial \boldsymbol{q}} = \mathrm{col}_i \left[\sum_j^n \sum_k^n \left(\frac{\partial M_{i,j}}{\partial q_k} - \frac{1}{2}\frac{\partial M_{j,k}}{\partial q_i}\right)\dot{q}_j \dot{q}_k\right] \tag{3.47}$$

is called centrifugal and Coriolis force vector (term), and $\boldsymbol{g}$ is called gravitational force vector (term).

Chapter 4

DYNAMICS BY RECURSIVE NEWTON EULER METHOD

Newton Euler Method calculates joint torques $\boldsymbol{\tau}$ using the joint trajectories $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\dot{\boldsymbol{q}}$ by recursive formulas. This section explains the basic idea and the procedure.

[Basic Idea 1]

$$\text{For each link, calculate} \left\{ \begin{array}{lcl} \boldsymbol{F}_i & = & m_i \ddot{\boldsymbol{x}}_i \\ \boldsymbol{N}_i & = & I_i \dot{\boldsymbol{\omega}}_i + \boldsymbol{\omega}_i \times I_i \boldsymbol{\omega}_i \end{array} \right.$$

However, interference forces and moments from other link makes difficult to find joint driving torques.

[Basic Idea 2]

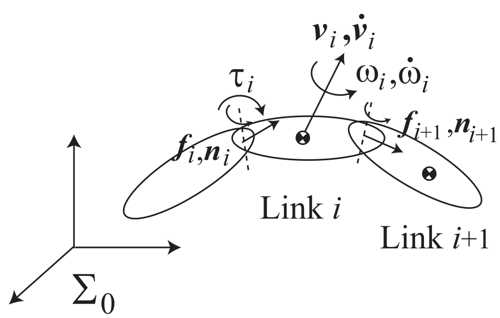


Fig. 4.1 Calculation by newton-euler method

(1) Calculate $\boldsymbol{v}_i$, $\boldsymbol{\omega}_i$ from $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$ ($i = 1 \rightarrow n$).

(2) Calculate $\boldsymbol{f}_i$, $\boldsymbol{n}_i$ at $i = n$. (Note that $\boldsymbol{f}_n$ and $\boldsymbol{n}_n$ are external force and moment on hand.)

(3) Calculate $\boldsymbol{f}_i$, $\boldsymbol{n}_i$ $(i = n - 1 \rightarrow 1)$ as reaction forces and moments.

4.1 Preliminaries of Newton Euler Method (Time Derivative of Rotation Matrix)

Recall Eq.(1.11),

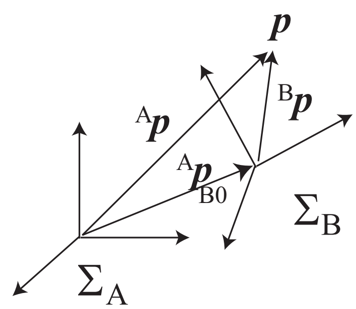


Fig. 4.2 Time derivative of position vector when the coordinate frame is rotating

$$^{A}\boldsymbol{p} = {}^{A}\boldsymbol{p}_{B0} + {}^{A}R_{B}\,{}^{B}\boldsymbol{p} \tag{4.1}$$

$$^{A}\dot{\boldsymbol{p}} = \frac{d}{dt}({}^{A}\boldsymbol{p}) = {}^{A}\dot{\boldsymbol{p}}_{B0} + \frac{d}{dt}({}^{A}R_{B})\,{}^{B}\boldsymbol{p} + {}^{A}R_{B}\,{}^{B}\dot{\boldsymbol{p}} \tag{4.2}$$

We here investigate second part of right hand.

$$\frac{d}{dt}({}^{A}R_{B}) = \frac{d}{dt}\left[{}^{A}\boldsymbol{x}_{B}\ {}^{A}\boldsymbol{y}_{B}\ {}^{A}\boldsymbol{z}_{B}\right] = \left[\frac{d}{dt}({}^{A}\boldsymbol{x}_{B})\ \frac{d}{dt}({}^{A}\boldsymbol{y}_{B})\ \frac{d}{dt}({}^{A}\boldsymbol{z}_{B})\right]$$

When the coordinate frame Σ_B rotates around vector ${}^{A}\boldsymbol{\omega}_{B}$, unit vector ${}^{A}\boldsymbol{x}_{B}$ also rotates around ${}^{A}\boldsymbol{\omega}_{B}$. Then

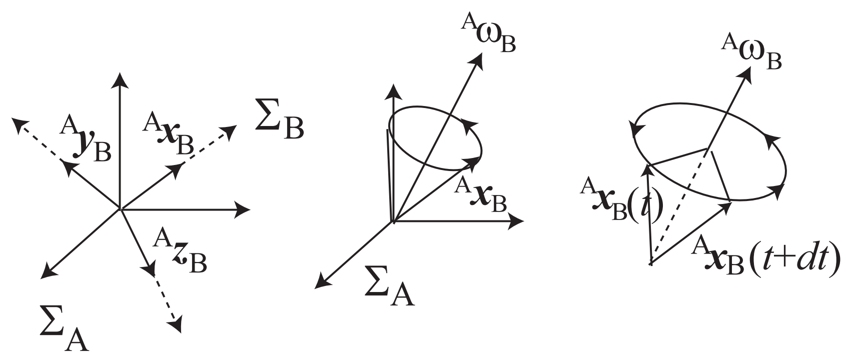


Fig. 4.3 Time derivative of rotation vector

the velocity of vector ${}^{A}\boldsymbol{x}_{B}$ is defined by

$$\frac{d{}^{A}\boldsymbol{x}_{B}}{dt} = \lim_{\Delta t \to 0} \frac{{}^{A}\boldsymbol{x}_{B}(t+\Delta t) - {}^{A}\boldsymbol{x}_{B}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta{}^{A}\boldsymbol{x}_{B}}{\Delta t} \tag{4.3}$$

From Fig.(4.3), the direction of vector $\Delta{}^{A}\boldsymbol{x}_{B}$ is perpendicular to the plane consisted with vectors ${}^{A}\boldsymbol{\omega}_{B}$ and ${}^{A}\boldsymbol{x}_{B}$. The sign is defined by right hand system with ${}^{A}\boldsymbol{\omega}_{B} \times {}^{A}\boldsymbol{x}_{B}$. The magnitude of vector $\Delta{}^{A}\boldsymbol{x}_{B}$ is

$$\left|\frac{d{}^{A}\boldsymbol{x}_{B}}{dt}\right|\Delta t = |{}^{A}\boldsymbol{\omega}_{B}|\Delta t|\sin\theta|$$

$$\left|\frac{d{}^{A}\boldsymbol{x}_{B}}{dt}\right| = |{}^{A}\boldsymbol{\omega}_{B}||\sin\theta| \tag{4.4}$$

As a result, we can describe the rotating vector of ${}^{A}\boldsymbol{x}_{B}$ by the following vector product

$$\frac{d{}^{A}\boldsymbol{x}_{B}}{dt} = {}^{A}\boldsymbol{\omega}_{B} \times {}^{A}\boldsymbol{x}_{B} \tag{4.5}$$

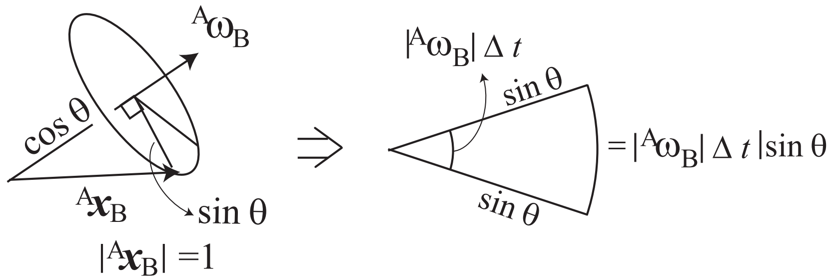


Fig. 4.4 Direction and magnitude of vector $\frac{d^A x_B}{dt}$

Combining other elements of $^A R_B$ leads to

$$\frac{d}{dt}(^A R_B) = \left[^A\boldsymbol{\omega}_B \times {}^A\boldsymbol{x}_B \;\; ^A\boldsymbol{\omega}_B \times {}^A\boldsymbol{y}_B \;\; ^A\boldsymbol{\omega}_B \times {}^A\boldsymbol{z}_B\right] \tag{4.6}$$

By using

$$\frac{d}{dt}(^A R_B)^B\boldsymbol{p} = {}^A\boldsymbol{\omega}_B \times {}^A R_B\, {}^B\boldsymbol{p}$$

the time derivative of vector $^A\boldsymbol{p}$ rotating around $^A\boldsymbol{\omega}_B$ is written by

$$^A\dot{\boldsymbol{p}} = {}^A\dot{\boldsymbol{p}}_{B0} + {}^A\boldsymbol{\omega}_B \times {}^A R_B\, {}^B\boldsymbol{p} + {}^A R_B {}^B\dot{\boldsymbol{p}} \tag{4.7}$$

We can calculate acceleration $^A\ddot{\boldsymbol{p}}$ using the same manner by

$$^A\ddot{\boldsymbol{p}} = {}^A\ddot{\boldsymbol{p}}_{B0} + {}^A\dot{\boldsymbol{\omega}}_B \times {}^A R_B\, {}^B\boldsymbol{p} + {}^A\boldsymbol{\omega}_B \times (^A\boldsymbol{\omega}_B \times {}^A R_B\, {}^B\boldsymbol{p}) + 2{}^A\boldsymbol{\omega}_B \times {}^A R_B\, {}^B\dot{\boldsymbol{p}} + {}^A R_B {}^B\ddot{\boldsymbol{p}} \tag{4.8}$$

4.2 Time Derivative of Angular Velocity

The angular velocity $^A\boldsymbol{\omega}_B$ is also a vector, thus we have the following relation between two coordinate frames

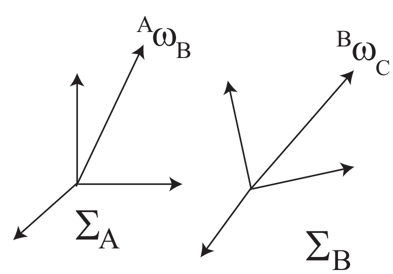


Fig. 4.5 Relation between two angular velocity

$$^A\boldsymbol{\omega}_C = {}^A\boldsymbol{\omega}_B + {}^A R_B\, {}^B\boldsymbol{\omega}_C \tag{4.9}$$

From Eq.(4.7),

$$^A\dot{\boldsymbol{\omega}}_C = {}^A\dot{\boldsymbol{\omega}}_B + {}^A\boldsymbol{\omega}_B \times {}^A R_B\, {}^B\boldsymbol{\omega}_C + {}^A R_B {}^B\dot{\boldsymbol{\omega}}_C \tag{4.10}$$

4.3 Basic Recursive Equation for Newton-Euler Method

In this recursive Newton-Euler method, following abbreviations are used.

R-joints: for rotational joints, T-joints: for translational joints

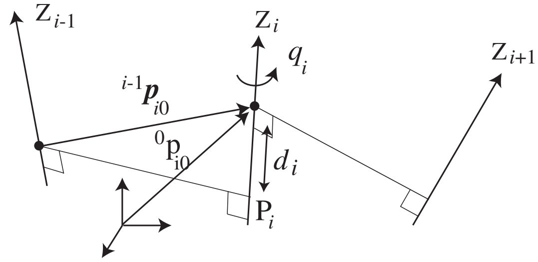


Fig. 4.6 Link coordinate systems

The angular velocity ${}^i\boldsymbol{\omega}_i$ is described by

$$ {}^i\boldsymbol{\omega}_i = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_i \end{bmatrix} \text{ (for R-joints)} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ (for T-joints)} \tag{4.11} $$

$$ \begin{aligned} {}^0\boldsymbol{\omega}_i &= {}^0\boldsymbol{\omega}_{i-1} + {}^0R_{i-1}\,{}^{i-1}\boldsymbol{\omega}_i \\ &= {}^0\boldsymbol{\omega}_{i-1} + {}^0R_i\,{}^i\boldsymbol{\omega}_i \\ &= {}^0\boldsymbol{\omega}_{i-1} + {}^0R_i \begin{bmatrix} 0 \\ 0 \\ \dot{q}_i \end{bmatrix} = {}^0\boldsymbol{\omega}_{i-1} + {}^0R_i\boldsymbol{z}\dot{q}_i \quad \text{(for R-joints)} \qquad (4.12) \\ &= {}^0\boldsymbol{\omega}_{i-1} \quad \text{(for T-joints)} \qquad (4.13) \end{aligned} $$

where

$$ \boldsymbol{z} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} $$

The derivative of ${}^0\boldsymbol{\omega}_i$ can be calculated by

$$ \begin{aligned} {}^0\dot{\boldsymbol{\omega}}_i &= {}^0\dot{\boldsymbol{\omega}}_{i-1} + {}^0\boldsymbol{\omega}_i \times {}^0R_i\boldsymbol{z}\dot{q}_i + {}^0R_i\boldsymbol{z}\ddot{q}_i \\ &= {}^0\dot{\boldsymbol{\omega}}_{i-1} + ({}^0\boldsymbol{\omega}_{i-1} + {}^0R_i\boldsymbol{z}\dot{q}_i) \times {}^0R_i\boldsymbol{z}\dot{q}_i + {}^0R_i\boldsymbol{z}\ddot{q}_i \\ &= {}^0\dot{\boldsymbol{\omega}}_{i-1} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_i\boldsymbol{z}\dot{q}_i + {}^0R_i\boldsymbol{z}\ddot{q}_i \quad \text{(for R-joints)} \qquad (4.14) \\ &= {}^0\dot{\boldsymbol{\omega}}_{i-1} \quad \text{(for T-joints)} \qquad (4.15) \end{aligned} $$

On the other hand, the origin of link coordinate frame ${}^0\boldsymbol{p}_i$ can be differentiated as followings

$$ \begin{aligned} {}^0\boldsymbol{p}_i &= {}^0\boldsymbol{p}_{i-1} + {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} \qquad (4.16) \\ {}^0\dot{\boldsymbol{p}}_i &= {}^0\dot{\boldsymbol{p}}_{i-1} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + {}^0R_{i-1}\,{}^{i-1}\dot{\boldsymbol{p}}_{i0} \\ &= {}^0\dot{\boldsymbol{p}}_{i-1} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + {}^0R_i\,{}^i\dot{\boldsymbol{p}}_{i0} \\ &= {}^0\boldsymbol{v}_i = {}^0\boldsymbol{v}_{i-1} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} \quad \text{(for R-joints)} \qquad (4.17) \\ &= {}^0\boldsymbol{v}_{i-1} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + {}^0R_{i-1}\boldsymbol{z}\dot{q}_i \quad \text{(for T-joints)} \qquad (4.18) \\ {}^0\dot{\boldsymbol{v}}_i &= {}^0\dot{\boldsymbol{v}}_{i-1} + {}^0\dot{\boldsymbol{\omega}}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} \quad \text{(for R-joints)} \qquad (4.19) \\ &= {}^0\dot{\boldsymbol{v}}_{i-1} + {}^0\dot{\boldsymbol{\omega}}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + {}^0\boldsymbol{\omega}_{i-1} \times {}^0\boldsymbol{\omega}_{i-1} \times {}^0R_{i-1}\,{}^{i-1}\boldsymbol{p}_{i0} + \\ &\quad 2\,{}^0\boldsymbol{\omega}_{i-1} \times {}^0R_i\boldsymbol{z}\dot{q}_i + {}^0R_i\boldsymbol{z}\ddot{q}_i \quad \text{(for T-joints)} \qquad (4.20) \end{aligned} $$

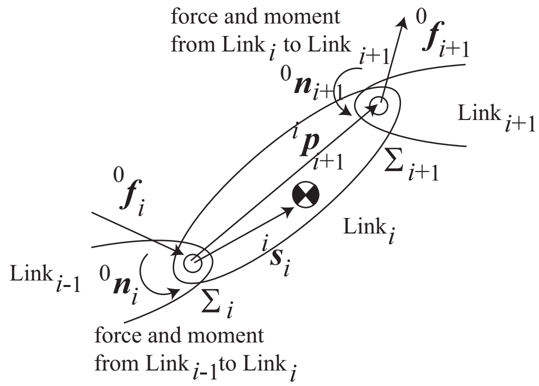


Fig. 4.7 Force and moment added from the other side link

4.4 Force and Moment Added to Links

Total force $^0\boldsymbol{F}_i$ and total moment $^0\boldsymbol{N}_i$ added with link i in Σ_0 coordinate frame is

$$\begin{aligned} {}^0\boldsymbol{F}_i &= {}^0\boldsymbol{f}_i - {}^0\boldsymbol{f}_{i+1} \qquad (4.21) \\ {}^0\boldsymbol{N}_i &= {}^0\boldsymbol{n}_i - {}^0\boldsymbol{n}_{i+1} + \text{ (arm vector)} \times {}^0\boldsymbol{f}_i - \text{ (arm vector)} \times {}^0\boldsymbol{f}_{i+1} \\ &= {}^0\boldsymbol{n}_i - {}^0\boldsymbol{n}_{i+1} - ({}^0R_i\,{}^i\boldsymbol{s}_i) \times {}^0\boldsymbol{f}_i - {}^0R_i({}^i\boldsymbol{p}_{i+1} - {}^i\boldsymbol{s}_i) \times {}^0\boldsymbol{f}_{i+1} \\ &= {}^0\boldsymbol{n}_i - {}^0\boldsymbol{n}_{i+1} - {}^0\hat{\boldsymbol{s}}_i \times {}^0\boldsymbol{f}_i - ({}^0\hat{\boldsymbol{p}}_{i+1} - {}^0\hat{\boldsymbol{s}}_i) \times {}^0\boldsymbol{f}_{i+1} \end{aligned}$$

where ${}^0\hat{\boldsymbol{s}}_i = {}^0R_i\,{}^i\boldsymbol{s}_i$ and ${}^0\hat{\boldsymbol{p}}_{i+1} = {}^0R_i\,{}^i\boldsymbol{p}_{i+1}$. By rewriting the equations into recursive forms,

$$\begin{aligned} {}^0\boldsymbol{f}_i &= {}^0\boldsymbol{F}_i + {}^0\boldsymbol{f}_{i+1} \qquad (4.22) \\ {}^0\boldsymbol{n}_i &= {}^0\boldsymbol{N}_i + {}^0\boldsymbol{n}_{i+1} + {}^0R_i\,{}^i\boldsymbol{s}_i \times {}^0\boldsymbol{F}_i + {}^0R_i\,{}^i\boldsymbol{p}_{i+1} \times {}^0\boldsymbol{f}_{i+1} \qquad (4.23) \end{aligned}$$

Considering the balance of force and moment by link motion and external force and moment,

$$\begin{aligned} {}^0\boldsymbol{F}_i &= m_i\,{}^0\ddot{\boldsymbol{s}}_i \qquad (4.24) \\ {}^0\boldsymbol{N}_i &= {}^0I_i\,{}^0\dot{\boldsymbol{\omega}}_i + {}^0\boldsymbol{\omega}_i \times {}^0I_i\,{}^0\boldsymbol{\omega}_i \qquad (4.25) \end{aligned}$$

where ${}^0\boldsymbol{s}_i$, ${}^0\dot{\boldsymbol{s}}_i$, ${}^0\ddot{\boldsymbol{s}}_i$ are

$$\begin{aligned} {}^0\boldsymbol{s}_i &= {}^0\boldsymbol{p}_i + {}^0R_i\,{}^i\boldsymbol{s}_i \qquad (4.26) \\ {}^0\dot{\boldsymbol{s}}_i &= {}^0\dot{\boldsymbol{p}}_i + {}^0\boldsymbol{\omega}_i \times {}^0R_i\,{}^i\boldsymbol{s}_i \qquad (4.27) \\ {}^0\ddot{\boldsymbol{s}}_i &= {}^0\ddot{\boldsymbol{p}}_i + {}^0\dot{\boldsymbol{\omega}}_i \times {}^0R_i\,{}^i\boldsymbol{s}_i + {}^0\boldsymbol{\omega}_i \times {}^0\boldsymbol{\omega}_i \times {}^0R_i\,{}^i\boldsymbol{s}_i \qquad (4.28) \end{aligned}$$

Note that there are no force and moment by gravitational force in the above equations. Those force and moment are considered later.

4.5 Formula of the Recursive Newton-Euler Method

Step 1) Set ${}^0\boldsymbol{\omega}_0$=${}^0\dot{\boldsymbol{\omega}}_0$=0, ${}^0\dot{\boldsymbol{v}}_0 = -\boldsymbol{g}$. (Note that this gravitational force condition affects all links.)

Step 2) Prepare m_i, ${}^i\boldsymbol{s}_i$, ${}^iI_i$, ${}^{i-1}T_i = \begin{bmatrix} {}^{i-1}R_i & {}^{i-1}\boldsymbol{p}_{i0} \\ 0 & 1 \end{bmatrix}$ for $i = 1, 2, \cdots, n$.

Give force and moment ${}^{n+1}\boldsymbol{f}_{n+1}$, ${}^{n+1}\boldsymbol{n}_{n+1}$ which is added to end-effector.

Step 3) Calculate ${}^i\boldsymbol{\omega}_i$, ${}^i\dot{\boldsymbol{\omega}}_i$, ${}^i\boldsymbol{v}_i$, ${}^i\ddot{\boldsymbol{s}}_i$ using the following equations for $i = 1 \to n$.

Multiplying ${}^iR_0$ with (4.12) and (4.13), we have

$$\begin{aligned} {}^iR_0\,{}^0\boldsymbol{\omega}_i = {}^i\boldsymbol{\omega}_i &= {}^iR_{i-1}\,{}^{i-1}\boldsymbol{\omega}_{i-1} + \boldsymbol{z}\dot{q}_i \quad \text{(for R-joints)} \qquad (4.29) \\ &= {}^iR_{i-1}\,{}^{i-1}\boldsymbol{\omega}_{i-1} \quad \text{(for T-joints)} \qquad (4.30) \end{aligned}$$

Multiplying ${}^iR_0$ with (4.14) and (4.15), we have

$$\begin{aligned} {}^i\dot{\boldsymbol{\omega}}_i &= {}^iR_{i-1}\,{}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} + {}^iR_{i-1}\,{}^{i-1}\boldsymbol{\omega}_{i-1} \times \boldsymbol{z}\dot{q}_i + \boldsymbol{z}\ddot{q}_i \quad \text{(for R-joints)} \qquad (4.31) \\ &= {}^iR_{i-1}\,{}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \quad \text{(for T-joints)} \qquad (4.32) \end{aligned}$$

Multiplying ${}^iR_0$ with (4.19) and (4.20), we have

$$\begin{aligned} {}^i\dot{\boldsymbol{v}}_i &= {}^iR_{i-1}\{{}^{i-1}\dot{\boldsymbol{v}}_{i-1} + {}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\boldsymbol{p}_{i0} + {}^{i-1}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\boldsymbol{p}_{i0}\} \qquad (4.33) \\ &\qquad \text{(for R-joints)} \\ &= {}^iR_{i-1}\{{}^{i-1}\dot{\boldsymbol{v}}_{i-1} + {}^{i-1}\dot{\boldsymbol{\omega}}_{i-1} \times {}^{i-1}\boldsymbol{p}_{i0} + {}^{i-1}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\boldsymbol{\omega}_{i-1} \times {}^{i-1}\boldsymbol{p}_{i0}\} + \\ &\quad 2{}^iR_{i-1}\,{}^{i-1}\boldsymbol{\omega}_{i-1} \times \boldsymbol{z}\dot{q}_i + \boldsymbol{z}\ddot{q}_i \quad \text{(for T-joints)} \qquad (4.34) \end{aligned}$$

Multiplying ${}^iR_0$ with (4.28), we have

$${}^i\ddot{\boldsymbol{s}}_i = {}^i\dot{\hat{\boldsymbol{v}}}_i = {}^i\dot{\boldsymbol{v}}_i + {}^i\dot{\boldsymbol{\omega}}_i \times {}^i\boldsymbol{s}_i + {}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{\omega}_i \times {}^i\boldsymbol{s}_i \quad \text{(for R and T-joints)} \qquad (4.35)$$

Step 4) Calculate ${}^i\boldsymbol{f}_i$, ${}^i\boldsymbol{n}_i$, $\boldsymbol{\tau}_i$ for $i = n \to 1$ (inversely) using the following equations.

Multiplying ${}^iR_0$ with (4.22) and (4.23), we have

$$\begin{aligned} {}^i\boldsymbol{f}_i &= m_i\,{}^i\ddot{\boldsymbol{s}}_i + {}^iR_{i+1}{}^{i+1}\boldsymbol{f}_{i+1} \qquad (4.36) \\ {}^i\boldsymbol{n}_i &= {}^iI_i\,{}^i\dot{\boldsymbol{\omega}}_i + {}^i\boldsymbol{\omega}_i \times {}^iI_i\,{}^i\boldsymbol{\omega}_i + {}^iR_{i+1}{}^{i+1}\boldsymbol{n}_{i+1} + \\ &\quad m_i\,{}^i\boldsymbol{s}_i \times {}^i\ddot{\boldsymbol{s}}_i + {}^i\boldsymbol{p}_{i+1} \times {}^iR_{i+1}\,{}^{i+1}\boldsymbol{f}_{i+1} \qquad (4.37) \end{aligned}$$

Using the above equations, joint torques $\boldsymbol{\tau}$ are calculated by

$$\begin{aligned} \boldsymbol{\tau}_i &= z \text{ element of } {}^i\boldsymbol{n}_i = (0\,0\,1) \cdot {}^i\boldsymbol{n}_i = \boldsymbol{z}_0^T\,{}^i\boldsymbol{n}_i \quad \text{(for R-joints)} \qquad (4.38) \\ &= z \text{ element of } {}^i\boldsymbol{f}_i = (0\,0\,1) \cdot {}^i\boldsymbol{f}_i = \boldsymbol{z}_0^T\,{}^i\boldsymbol{f}_i \quad \text{(for T-joints)} \qquad (4.39) \end{aligned}$$

where we use the relation

$$ {}^iI_i = ({}^0R_i)^T\, {}^0I_i\, ({}^0R_i), \quad {}^0I_i = {}^0R_i\, {}^iI_i\, ({}^0R_i)^T \quad \text{(see Appendix D)} $$

and ${}^{i-1}\boldsymbol{p}_{i0}$ can be described by

$$ {}^{i-1}\boldsymbol{p}_{i0} = [a_i \quad -d_i \sin\alpha_i \quad d_i \cos\alpha_i]^T \tag{4.40} $$

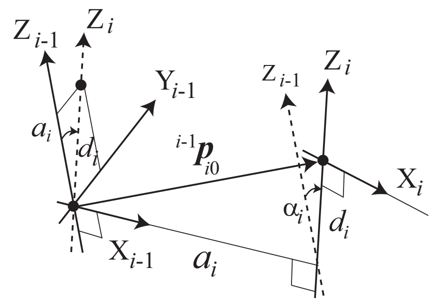


Fig. 4.8 Elements of ${}^{i-1}p_{i0}$

Chapter 5

FORWARD DYNAMICS AND INVERSE DYNAMICS

5.1 Inverse Dynamics

The inverse dynamics is represented by

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) = \boldsymbol{\tau} \tag{5.1}$$

The equation calculates joint torque $\boldsymbol{\tau}$ for given joint trajectory $\boldsymbol{q}$, $\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{q}}$.

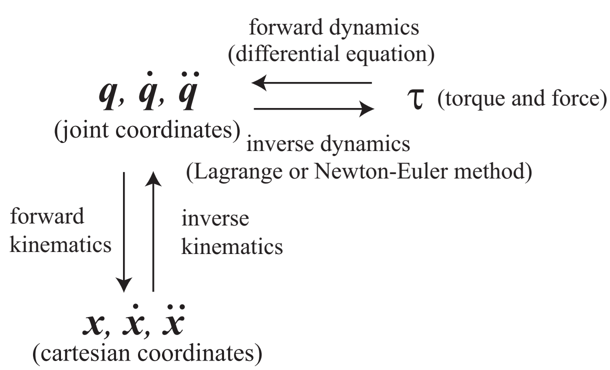


Fig. 5.1 Forward and inverse dynamics

5.2 Forward Dynamics

When we simulate the dynamics of manipulator, we need forward dynamics calculation. By pre-multiplying inverse of inertia moment matrix M to Eq.(5.1),

$$\ddot{\boldsymbol{q}} = M^{-1}(\boldsymbol{q}) \left[\boldsymbol{\tau} - \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \boldsymbol{g}(\boldsymbol{q})\right] \tag{5.2}$$

Note that matrix M is positive definite. Using the notation of

$$\begin{aligned} [x_1, x_2, \cdots, x_n]^T &= [q_1, q_2, \cdots, q_n]^T \\ [x_{n+1}, x_{n+2}, \cdots, x_{2n}]^T &= [\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n]^T \end{aligned}$$

the differential equation (5.2) is rewritten as

$$\left\{\begin{array}{ccccccc} \dot{x}_1 & = & x_{n+1} & & & & \\ \vdots & & \vdots & & & & \\ \dot{x}_n & = & x_{2n} & & & & \\ \dot{x}_{n+1} & = & \ddot{q}_1 & = & \{M^{-1}(\boldsymbol{q})\,[\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}]\}_1 & = & f_{n+1}(\boldsymbol{x},\boldsymbol{\tau}) \\ \vdots & & \vdots & & \vdots & & \vdots \\ \dot{x}_{2n} & = & \ddot{q}_n & = & \{M^{-1}(\boldsymbol{q})\,[\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}})+\boldsymbol{g}(\boldsymbol{q})-\boldsymbol{\tau}]\}_n & = & f_{2n}(\boldsymbol{x},\boldsymbol{\tau}) \end{array}\right. \tag{5.3}$$

As a result the differential equation representing dynamics can be represented by the form of

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x},\boldsymbol{\tau}) \quad (\boldsymbol{x} \in R^{2n}) \tag{5.4}$$

The forward dynamics calculation is, then to solve the above differential equation with initial condition ($\boldsymbol{x}(0) = [\boldsymbol{q}(0), \dot{\boldsymbol{q}}(0)]^T$) and input $\boldsymbol{\tau}(t)$ $(0 \le t \le t_f)$. This can be solved by numerically (for example by Runge-Kutta method).

Chapter 6

CONTROL

The actual robotic arm is usually driven by DC or AC servo motors. For the discussion of the robotic control, we need some mathematical model of the "mechanical part" and the "electrical part" of the robotics system. The mathematical model of the mechanical part is given by (3.29). We now need the model of electrical part which is the model of robotic actuator. As the mathematical model of the actuator, DC servo motor is explained. The mathematical model of AC servo motor is almost same, which is omitted in this textbook. After the modeling of the actuator, two models of robotic arm and actuator part including gear train are combined as a model of robotic system to design control laws.

6.1 Modeling of Actuator and Transmission Mechanism

In this modeling actuator part, we assume that DC motor and gear train is used for driving mechanism of robotic link. Followings are nomenclature for the modeling.

v_M: added voltage for DC motor (V)

R_M: armature resistance of DC motor (Ω)

L_M: armature inductance of DC motor (H)

i_M: armature current of DC motor (A)

q_M: rotation angle of DC motor axis (rad)

K_e: inverse electromotive force constant of DC motor (V·s/rad)

K_M: torque constant of DC motor (Kgm/A)

τ_0: generated torque of motor (Kgm^2/s^2)

J_M: inertia moment of motor axis and pinion gear

τ_M: output axis torque

Considering voltage drop in Fig.(6.1) circuit,

$$v_M = R_M i_M + K_e \dot{q}_M + L_M \frac{di_M}{dt} \tag{6.1}$$

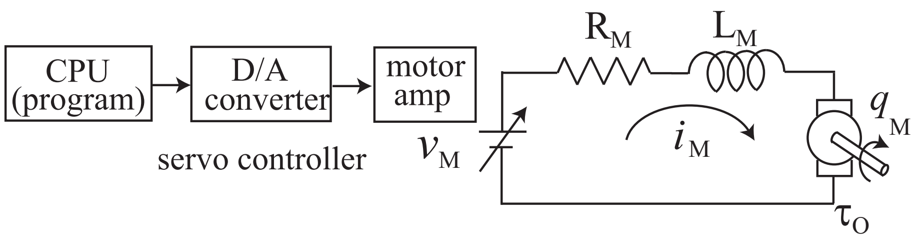


Fig. 6.1 Model of DC motor

Since armature inductance is small for normal DC motor,

$$v_M = R_M i_M + K_e \dot{q}_M \tag{6.2}$$

Because of structure of DC motor

$$\begin{aligned} \tau_O &= k_M i_M & (6.3) \\ &= J_M \ddot{q}_M + \tau_M & (6.4) \end{aligned}$$

For normal DC motor, the speed is too high and torque is too small to drive robot arms. Thus most robot arms

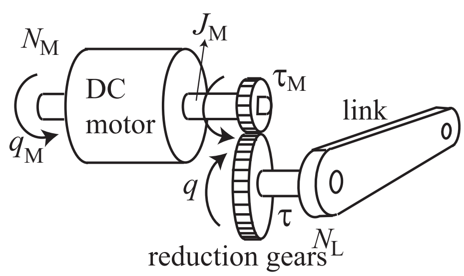


Fig. 6.2 Model of gear train and link

has reduction gears in its joint. The reduction ratio of the gear train is defined by

$$\text{reduction ratio} = \frac{\text{revolving speed of output shaft}}{\text{revolving speed of input shaft}} = \frac{N_L}{N_M} \quad (\leq 1 \text{ for most robot}) \tag{6.5}$$

Or, gear ratio is defined by the inverse.

$$\text{gear ratio} = \frac{\text{number of tooth of output gear}}{\text{number of tooth of input gear}} = \frac{1}{\text{reduction ratio}} = \gamma \quad (\geq 1 \text{ for most robot}) \tag{6.6}$$

Using the definition, we have the relation of δq_M and δq

$$\delta q_M = \gamma \delta q \tag{6.7}$$

By collecting all n-joints

$$\delta \boldsymbol{q}_M = \Gamma \delta \boldsymbol{q} \tag{6.8}$$

where $\Gamma = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{bmatrix}$. By principle of virtual work

$$\boldsymbol{\tau}_M^T \delta \boldsymbol{q}_M = \boldsymbol{\tau}^T \delta \boldsymbol{q} \tag{6.9}$$

Using Eq.(6.8),

$$\boldsymbol{\tau}_M^T \Gamma \delta \boldsymbol{q} = \boldsymbol{\tau}^T \delta \boldsymbol{q}$$

By taking transpose for both sides,

$$\delta \boldsymbol{q}^T \Gamma^T \boldsymbol{\tau}_M = \delta \boldsymbol{q}^T \boldsymbol{\tau}^T$$

Then we have a relation between motor torque and joint output torque,

$$\boldsymbol{\tau} = \Gamma^T \boldsymbol{\tau}_M \tag{6.10}$$

We see that the output torque is multiplied by γ from motor axis. We next derive a dynamics equation in which input is motor voltage. At first, from Eq.(6.3),

$$\boldsymbol{\tau}_O = \hat{K}_M \boldsymbol{i}_M \tag{6.11}$$

where $\hat{K}_M = \begin{bmatrix} K_{M1} & & 0 \\ & \ddots & \\ 0 & & K_{Mn} \end{bmatrix}$. Similar notations are used for R_M, K_e and J_M. By substituting $\boldsymbol{i}_M = \hat{K}_M^{-1} \boldsymbol{\tau}_O$ (from (6.11)) into Eq.(6.2)

$$\boldsymbol{v}_M = \hat{R}_M \hat{K}_M^{-1} \boldsymbol{\tau}_O + \hat{K}_e \dot{\boldsymbol{q}}_M \tag{6.12}$$

From Eq.(6.7), $\dfrac{\delta q_M}{\delta t} = \gamma \dfrac{\delta q}{\delta t}$. Thus we have $\dot{\boldsymbol{q}}_M = \Gamma \dot{\boldsymbol{q}}$. By substituting the equation into Eq.(6.12) and solving with $\boldsymbol{\tau}_O$,

$$\boldsymbol{\tau}_O = \hat{K}_M \hat{R}_M^{-1} (\boldsymbol{v}_M - \hat{K}_e \Gamma \dot{\boldsymbol{q}}) \tag{6.13}$$

On the other hand, from Eq.(6.4) and Eq.(6.10)

$$\boldsymbol{\tau}_O = \hat{J}_M \ddot{\boldsymbol{q}}_M + \boldsymbol{\tau}_M = \hat{J}_M \Gamma \ddot{\boldsymbol{q}} + \Gamma^{-1} \boldsymbol{\tau} \tag{6.14}$$

By setting Eq.(6.13) = Eq.(6.14) and solving with $\boldsymbol{\tau}$

$$\boldsymbol{\tau} = \Gamma^T \hat{K}_M \hat{R}_M^{-1} \boldsymbol{v}_M - \Gamma^T \hat{K}_M \hat{R}_M^{-1} \hat{K}_e \Gamma \dot{\boldsymbol{q}} - \Gamma^T \hat{J}_M \Gamma \ddot{\boldsymbol{q}} \tag{6.15}$$

From the result in the section of DYNAMICS

$$\boldsymbol{\tau} = M(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + D \dot{\boldsymbol{q}} \tag{6.16}$$

where we add viscous friction coefficient matrix D to the dynamics equation. From Eq.(6.15) and Eq.(6.16),

$$M'(\boldsymbol{q}) \ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + D' \dot{\boldsymbol{q}} = \hat{K} \boldsymbol{v}_M \tag{6.17}$$

where $\left\{ \begin{array}{lcl} M' & = & M(\boldsymbol{q}) + \Gamma^T \hat{J}_M \Gamma \\ D' & = & D + \Gamma^T \hat{K}_M \hat{R}_M^{-1} \hat{K}_e \Gamma \\ \hat{K} & = & \Gamma^T \hat{K}_M \hat{R}_M^{-1} \end{array} \right.$.

6.2 Control of Robot Arm

Following various control methods are used in the industrial robots or proposed.

(a) PD(PID) control for each joint

(b) PD(PID) control with gravitational force compensation for each joint

(c) Computed torque method

(d) Resolved acceleration method

(e) Force control

(f) Other control method (Adaptive control, Learning control, Neural and Fuzzy control)

6.3 PD Controller for Each Joint

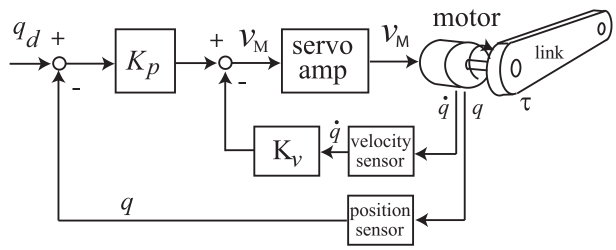


Fig. 6.3 PD controller

The PD controller which feedbacks position error and velocity for each joint is described by

$$\boldsymbol{v}_M = -\hat{K}_v \dot{\boldsymbol{q}} + \hat{K}_p(\boldsymbol{q} - \boldsymbol{q}_d) \tag{6.18}$$

where $\hat{K}_p = \begin{bmatrix} K_{p1} & & 0 \\ & \ddots & \\ 0 & & K_{pn} \end{bmatrix}$, $\hat{K}_v = \begin{bmatrix} K_{v1} & & 0 \\ & \ddots & \\ 0 & & K_{vn} \end{bmatrix}$, $\boldsymbol{q}_d$ is desired joint position. In the followings, we analyze response characteristics for the controler. From Eq.(6.17) and Eq.(6.18),

$$-(M(\boldsymbol{q})+\Gamma^T \hat{J}_M \Gamma)\ddot{\boldsymbol{q}} - \left[\Gamma^T \hat{K}_M \hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + D\right]\dot{\boldsymbol{q}} + \Gamma^T \hat{K}_M \hat{R}_M^{-1}\hat{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) = \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \tag{6.19}$$

By denoting $\boldsymbol{e} = \boldsymbol{q}_d - \boldsymbol{q}$, $\dot{\boldsymbol{e}} = -\dot{\boldsymbol{q}}$, $\ddot{\boldsymbol{e}} = -\ddot{\boldsymbol{q}}$,

$$(M(\boldsymbol{q}) + \Gamma^T \hat{J}_M \Gamma)\ddot{\boldsymbol{e}} - \left[\Gamma^T \hat{K}_M \hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + D\right]\dot{\boldsymbol{e}} + \Gamma^T \hat{K}_M \hat{R}_M^{-1}\hat{K}_p \boldsymbol{e} = \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) \tag{6.20}$$

If the reduction ratio is big and joint velocity is small, then we can neglect gravitational force and $M(\boldsymbol{q}) \Rightarrow 0$, $\boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \Rightarrow 0$, $\boldsymbol{g}(\boldsymbol{q}) \Rightarrow 0$. For such case, the error equation is

$$\ddot{\boldsymbol{e}} + \left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M \hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T \hat{J}_M \Gamma)^{-1} D\right]\dot{\boldsymbol{e}} + (\hat{J}_M\Gamma)^{-1}\hat{K}_M \hat{R}_M^{-1}\hat{K}_p \boldsymbol{e} = 0 \tag{6.21}$$

This equation is independent quadratic system for each joint, because $\hat{J}_M$, Γ, $\hat{K}_M$, $\hat{R}_M$, $\hat{K}_e$, $\hat{K}_v$, $\hat{K}_p$ are all diagonal matrices. Thus, each element of Eq.(6.21) is described by

$$\ddot{e} + k_v \dot{e} + k_p e = 0 \tag{6.22}$$

By setting appropriate k_p and k_v, we can realize desired response of joint angle.

6.4 PD Controller Analysis Considering Gravitational Force

When the gravitational force term can not be neglected, the error equation is represented from Eq.(6.20) by

$$\ddot{\boldsymbol{e}} + \left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M \hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T \hat{J}_M \Gamma)^{-1} D\right]\dot{\boldsymbol{e}} + (\hat{J}_M\Gamma)^{-1}\hat{K}_M \hat{R}_M^{-1}\hat{K}_p \boldsymbol{e} = (\Gamma^T \hat{J}_M \Gamma)^{-1}\boldsymbol{g}(\boldsymbol{q}) \tag{6.23}$$

The gravitational force term is basically non-linear term. It makes difficult to analyze further. Then we here only consider neighborhood of $\boldsymbol{q}_d$. By expanding $\boldsymbol{g}(\boldsymbol{q})$ of Eq(6.23) at $\boldsymbol{q} = \boldsymbol{q}_d$ and taking until first order term, then we have

$$\text{right hand side of Eq.(6.23)} = (\Gamma^T \hat{J}_M \Gamma)^{-1}\left\{\boldsymbol{g}(\boldsymbol{q}_d) + \left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right]_{\boldsymbol{q}=\boldsymbol{q}_d}(\boldsymbol{q} - \boldsymbol{q}_d)\right\} \tag{6.24}$$

By representing $\left[\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{q}}\right]_{\boldsymbol{q}=\boldsymbol{q}_d} = C$ (constant matrix) and doing Laplace transformation

$$s^2\boldsymbol{e}(s) + s\left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T\hat{J}_M\Gamma)^{-1}D\right]\boldsymbol{e}(s) + (\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p\boldsymbol{e}(s)$$

$$= (\Gamma^T\hat{J}_M\Gamma)^{-1}\left\{\frac{\boldsymbol{g}(\boldsymbol{q}_d)}{s} - C\boldsymbol{e}(s)\right\} \tag{6.25}$$

Solving the equation with $\boldsymbol{e}(s)$ leads to

$$\boldsymbol{e}(s) = \left\{s^2E_n + s\left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T\hat{J}_M\Gamma)^{-1}D\right] + \right.$$

$$\left.(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p + (\Gamma^T\hat{J}_M\Gamma)^{-1}C\right\}^{-1}(\Gamma^T\hat{J}_M\Gamma)^{-1}\left\{\frac{\boldsymbol{g}(\boldsymbol{q}_d)}{s}\right\} \tag{6.26}$$

We apply final value theorem of Laplace transformation for the equation.

$$\lim_{t\to\infty}\boldsymbol{e}(t) = \lim_{s\to 0}s\boldsymbol{e}(s) = \left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p + (\Gamma^T\hat{J}_M\Gamma)^{-1}C\right]^{-1}(\Gamma^T\hat{J}_M\Gamma)^{-1}\boldsymbol{g}(\boldsymbol{q}_d) \tag{6.27}$$

We see that offset remains.

6.5 PID controler for Each Joint

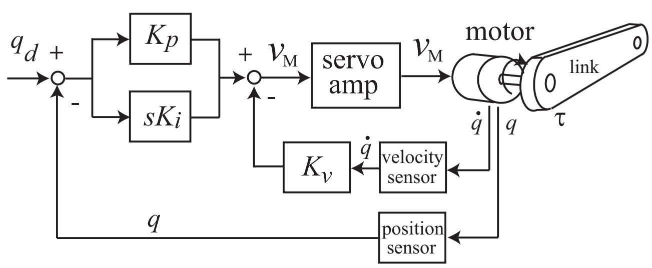


Fig. 6.4 PID controler

The PID controler is given by

$$\boldsymbol{v}_M = -\hat{K}_v\dot{\boldsymbol{q}} + \hat{K}_p(\boldsymbol{q} - \boldsymbol{q}_d) + \hat{K}_i\int(\boldsymbol{q} - \boldsymbol{q}_d)dt \tag{6.28}$$

By setting $\boldsymbol{e} = \boldsymbol{q} - \boldsymbol{q}_d$, the error equation for the controler is

$$\ddot{\boldsymbol{e}} + \left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T\hat{J}_M\Gamma)^{-1}D\right]\dot{\boldsymbol{e}} + (\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p\boldsymbol{e} +$$

$$(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_i\int\boldsymbol{e}dt = (\Gamma^T\hat{J}_M\Gamma)^{-1}\boldsymbol{g}(\boldsymbol{q}) \tag{6.29}$$

By linearizing the gravitational term similarly in the previous section,

$$\text{right hand side of Eq.(6.29)} = (\Gamma^T\hat{J}_M\Gamma)^{-1}\left\{\boldsymbol{g}(\boldsymbol{q}_d) + C(\boldsymbol{q} - \boldsymbol{q}_d)\right\} \tag{6.30}$$

Using Laplace transformation,

$$s^2\boldsymbol{e}(s) + s\left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T\hat{J}_M\Gamma)^{-1}D\right]\boldsymbol{e}(s) + (\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p\boldsymbol{e}(s) +$$

$$\frac{1}{s}(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_i\boldsymbol{e}(s) = (\Gamma^T\hat{J}_M\Gamma)^{-1}\left\{\frac{\boldsymbol{g}(\boldsymbol{q}_d)}{s} - C\boldsymbol{e}(s)\right\} \tag{6.31}$$

By solving with $\boldsymbol{e}(s)$,

$$\boldsymbol{e}(s) = \left\{s^3E_n + s^2\left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v) + (\Gamma^T\hat{J}_M\Gamma)^{-1}D\right] + s\left[(\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\hat{K}_p + (\Gamma^T\hat{J}_M\Gamma)^{-1}C\right] + (\hat{J}_M\Gamma)^{-1}\hat{K}_M\hat{R}_M^{-1}\right\}^{-1}(\Gamma^T\hat{J}_M\Gamma)^{-1}\boldsymbol{g}(\boldsymbol{q}_d) \tag{6.32}$$

From final value theorem,

$$\lim_{t\to\infty}\boldsymbol{e}(t) = 0 \tag{6.33}$$

We see that the PID controller has no offset provided that $\boldsymbol{q}$ is near $\boldsymbol{q}_d$.

6.6 PD Controller with Gravitational Force Compensation

We here consider the following controller which is PD controller with gravitational compensation.

$$\boldsymbol{v}_M = -\hat{K}_v\dot{\boldsymbol{q}} + \hat{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) + \hat{R}_M\hat{K}_M^{-1}(\Gamma^T)^{-1}\boldsymbol{g}(\boldsymbol{q}) \tag{6.34}$$

Note that this is a non-linear controller. From Eq.(6.19) and Eq.(6.34) ($D = 0$ for simplicity)

$$(M(\boldsymbol{q}) + \Gamma^T\hat{J}_M\Gamma)\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \Gamma^T\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v)\dot{\boldsymbol{q}} + \Gamma^T\hat{K}_M\hat{R}_M^{-1}\hat{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) = 0 \tag{6.35}$$

As seen in the previous discussion, this control system is quadratic system provided that reduction ratio is big and joint velocity is small. However, in this section, we analyze a stability of the control system without such approximation or assumption. At first, we select the following function as a candidate of Lyapunov function,

$$V(t) = \frac{1}{2}\left\{\dot{\boldsymbol{q}}^T(M(\boldsymbol{q}) + \Gamma^T\hat{J}_M\Gamma)\dot{\boldsymbol{q}} + (\boldsymbol{q} - \boldsymbol{q}_d)^T\Gamma^T\hat{K}_M\hat{R}_M^{-1}\hat{K}_p(\boldsymbol{q} - \boldsymbol{q}_d)\right\} \tag{6.36}$$

$M(\boldsymbol{q}) + \Gamma^T\hat{J}_M\Gamma$ and $\Gamma^T\hat{K}_M\hat{R}_M^{-1}\hat{K}_p$ are both positive definite matrix. Thus $V(t) > 0$. The time derivative of $V(t)$ is

$$\begin{aligned}\dot{V}(t) &= \dot{\boldsymbol{q}}^T\left\{(M(\boldsymbol{q}) + \Gamma^T\hat{J}_M\Gamma)\ddot{\boldsymbol{q}} + \frac{1}{2}\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} + \Gamma^T\hat{K}_M\hat{R}_M^{-1}\hat{K}_p(\boldsymbol{q} - \boldsymbol{q}_d)\right\}\\ &= \dot{\boldsymbol{q}}^T\left\{-\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) + \frac{1}{2}\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\right\} - \dot{\boldsymbol{q}}^T\Gamma^T\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v)\dot{\boldsymbol{q}}\end{aligned} \tag{6.37}$$

Where

$$\begin{aligned}\dot{\boldsymbol{q}}^T\boldsymbol{h}(\boldsymbol{q},\dot{\boldsymbol{q}}) &= \dot{\boldsymbol{q}}^T\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}^T\frac{\partial}{\partial\boldsymbol{q}}(\frac{1}{2}\dot{\boldsymbol{q}}^TM(\boldsymbol{q})\dot{\boldsymbol{q}})\\ &= \dot{\boldsymbol{q}}^T\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\sum_{i=1}^{n}\frac{\partial}{\partial q_i}(\dot{\boldsymbol{q}}^TM(\boldsymbol{q})\dot{\boldsymbol{q}})\dot{q}_i\\ &= \dot{\boldsymbol{q}}^T\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}} - \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\\ &= \frac{1}{2}\dot{\boldsymbol{q}}^T\dot{M}(\boldsymbol{q})\dot{\boldsymbol{q}}\end{aligned} \tag{6.38}$$

Using the relation,

$$\dot{V}(t) = \dot{\boldsymbol{q}}^T\Gamma^T\hat{K}_M\hat{R}_M^{-1}(\hat{K}_e\Gamma + \hat{K}_v)\dot{\boldsymbol{q}} \le 0 \tag{6.39}$$

Thus, $V(t)$ is a Lyapunov function. Equality is satisfied when $\dot{\boldsymbol{q}}(t) = 0$, where $\boldsymbol{q}(t) = \boldsymbol{q}_d$. By the above discussion, if $\boldsymbol{q}(t) \neq \boldsymbol{q}_d$, then $\dot{V}(t) < 0$. Therefore the control system Eq.(6.34) is asymptotically stable to $\boldsymbol{q}_d$.

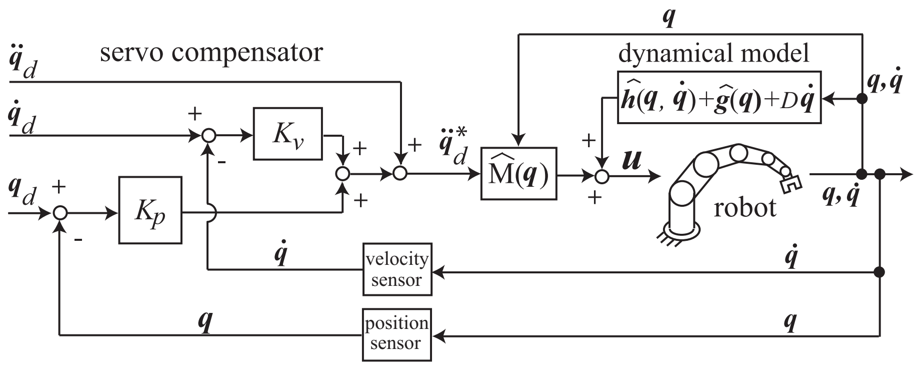


Fig. 6.5 Computed torque method

6.7 Computed Torque Method

The computed torque method is a PD (PID) controller with robot dynamics compensation. The nonlinear dynamics is calculated, then the controller is linearized. We here describe the robot dynamics by

$$M(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \boldsymbol{g}(\boldsymbol{q}) + D\dot{\boldsymbol{q}} = \boldsymbol{u} \tag{6.40}$$

where $\boldsymbol{u}$ is input vector (torque $\boldsymbol{\tau}$ or motor voltage $\boldsymbol{v}$). The control law of computed torque method is represented by

$$\boldsymbol{u} = \hat{M}(\boldsymbol{q})\ddot{\boldsymbol{q}}^* + \hat{\boldsymbol{h}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \hat{D}\dot{\boldsymbol{q}} \tag{6.41}$$

$$\ddot{\boldsymbol{q}}^* = \ddot{\boldsymbol{q}}_d(t) + \hat{K}_v(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \hat{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) \tag{6.42}$$

where

$$\begin{cases} \hat{M}(\boldsymbol{q}): & \text{model of inertia matrix} \\ \hat{\boldsymbol{h}}(\boldsymbol{q}, \dot{\boldsymbol{q}}): & \text{model of centrifugal and Coriolis force} \\ \hat{\boldsymbol{g}}(\boldsymbol{q}): & \text{model of gravitational force} \\ \hat{D}: & \text{model of viscous friction coefficient} \end{cases}$$

If model is accurate,

$$\hat{M}(\boldsymbol{q}) = M(\boldsymbol{q}), \;\; \hat{\boldsymbol{h}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \boldsymbol{h}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \;\; \hat{\boldsymbol{g}}(\boldsymbol{q}) = \boldsymbol{g}(\boldsymbol{q}), \;\; \hat{D} = D \tag{6.43}$$

then, substituting Eq.(6.41), (6.42), (6.43) into (6.40)

$$\ddot{\boldsymbol{q}}^* = \ddot{\boldsymbol{q}} \tag{6.44}$$

Then, from Eq.(6.42) and Eq.(6.44),

$$\ddot{\boldsymbol{q}}_d(t) - \ddot{\boldsymbol{q}} + \hat{K}_v(\dot{\boldsymbol{q}}_d - \dot{\boldsymbol{q}}) + \hat{K}_p(\boldsymbol{q}_d - \boldsymbol{q}) = 0 \tag{6.45}$$

Thus, error equation is

$$\ddot{\boldsymbol{e}} + \hat{K}_v\dot{\boldsymbol{e}} + \hat{K}_p\boldsymbol{e} = 0 \tag{6.46}$$

By selecting $\hat{K}_v$ and $\hat{K}_p$ properly, we can realize desirable response of arm motion.

6.8 PD(PID) Feedback in Workspace Coordinates

Consider the case that the hand position of robot should be controlled for an object fixed with workspace coordinate frame, such as welding work in which welding seam line is described by workspace coordinate frame. For such case, the deviation of hand position in workspace coordinates should be feedbacked. One of such control law is

$$\boldsymbol{u} = J_\omega^T(\boldsymbol{q})\hat{K}_p(\boldsymbol{r}_d - \boldsymbol{r}) - \hat{K}_v\dot{\boldsymbol{q}} + \boldsymbol{g}(\boldsymbol{q}) \tag{6.47}$$

The stability of the control law is also guaranteed by the similar way in the section of PD control with gravitational force compensation.

6.9 Resolved Acceleration Control Method

The control law of resolved acceleration method is given by

$$\boldsymbol{u} = \hat{M}(\boldsymbol{q})J^{-1}(\boldsymbol{q})(\ddot{\boldsymbol{r}}^* - \dot{J}(\boldsymbol{q})\dot{\boldsymbol{q}}) + \hat{\boldsymbol{h}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) + \hat{\boldsymbol{g}}(\boldsymbol{q}) + \hat{D}\dot{\boldsymbol{q}} \tag{6.48}$$

$$\ddot{\boldsymbol{r}}^* = \ddot{\boldsymbol{r}}_d(t) + \hat{K}_v(\dot{\boldsymbol{r}}_d - \dot{\boldsymbol{r}}) + \hat{K}_p(\boldsymbol{r}_d - \boldsymbol{r}) \tag{6.49}$$

This control law is work space feedback type with dynamics compensation, whereas the computed torque method is joint space feedback type. Similarly with computed torque method, if $\ddot{\boldsymbol{r}} = \ddot{\boldsymbol{r}}^*$ and model of dynamical parameter is accurate, then we have same error equation as Eq.(6.46).

Appendix A

Formula of Vector Product

A.1 Vector product

Definition of vector product.

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix} \tag{A.1}$$

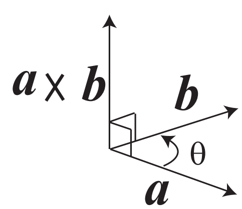


Fig. A.1 Definition of vector product

$$|\boldsymbol{a} \times \boldsymbol{b}| = |\boldsymbol{a}|\ |\boldsymbol{b}|\ \sin\theta \tag{A.2}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \tag{A.3}$$

$$\boldsymbol{a}^T(\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \text{(scalar value)} \tag{A.4}$$

$$\boldsymbol{a}^T(\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}^T(\boldsymbol{c} \times \boldsymbol{a}) = \boldsymbol{c}^T(\boldsymbol{a} \times \boldsymbol{b}) \tag{A.5}$$

A.2 Vector Triple Product

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_y c_z - b_z c_y & b_z c_x - b_x c_z & b_x c_y - b_y c_x \end{vmatrix} = \begin{bmatrix} a_y(b_x c_y - b_y c_x) - a_z(b_y c_z - b_z c_y) \\ a_z(b_y c_z - b_z c_y) - a_x(b_x c_y - b_y c_x) \\ a_x(b_z c_x - b_x c_z) - a_y(b_y c_z - b_z c_y) \end{bmatrix} \tag{A.6}$$

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b} - (\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} \tag{A.7}$$

$$(\boldsymbol{c}^T\boldsymbol{a})\boldsymbol{b} = (\boldsymbol{c}^T\boldsymbol{a})E_3\boldsymbol{b} \tag{A.8}$$

$$(\boldsymbol{a}^T\boldsymbol{b})\boldsymbol{c} = (\boldsymbol{c}\boldsymbol{a}^T)\boldsymbol{b} \tag{A.9}$$

From Eq.(A.7), Eq.(A.8), Eq.(A.9),

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\boldsymbol{c}^T \boldsymbol{a})\boldsymbol{b} - (\boldsymbol{a}^T \boldsymbol{b})\boldsymbol{c} \\ &= (\boldsymbol{c}^T \boldsymbol{a})E_3\boldsymbol{b} - (\boldsymbol{c}\boldsymbol{a}^T)\boldsymbol{b} \\ &= (\boldsymbol{c}^T \boldsymbol{a}E_3 - \boldsymbol{c}\boldsymbol{a}^T)\boldsymbol{b} \end{aligned} \tag{A.10}$$

This is commutative formula between vector triple product and matrix times vector.

Appendix B

Rotation Matrix for Arbitrary Axis

We derive the rotation matrix rotated by α around arbitrary axis $\boldsymbol{k}$:

$$R = Rot(\boldsymbol{k}, \alpha)$$

where $\boldsymbol{k}$ is unit vector. Consider unit vector $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ and vector $\boldsymbol{k}$ as in Fig.(B.1). The vector $\boldsymbol{p}$ can be described by

$$\boldsymbol{p} = (\boldsymbol{p}^T\boldsymbol{i})\boldsymbol{i} + (\boldsymbol{p}^T\boldsymbol{j})\boldsymbol{j} + (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k} \tag{B.1}$$

Consider $\boldsymbol{i}^*$ which is obtained by rotating the vector $\boldsymbol{i}$ around $\boldsymbol{k}$ with angle α,

$$\boldsymbol{i}^* = \boldsymbol{i}\cos\alpha + \boldsymbol{j}\sin\alpha \tag{B.2}$$

then $\boldsymbol{j}^*$ is

$$\boldsymbol{j}^* = -\boldsymbol{i}\sin\alpha + \boldsymbol{j}\cos\alpha \tag{B.3}$$

The vector $\boldsymbol{p}$ is also rotated with α. The rotated vector is denoted by $\boldsymbol{p}^*$ which is

$$\boldsymbol{p}^* = (\boldsymbol{p}^{*T}\boldsymbol{i}^*)\boldsymbol{i}^* + (\boldsymbol{p}^{*T}\boldsymbol{j}^*)\boldsymbol{j}^* + (\boldsymbol{p}^{*T}\boldsymbol{k}^*)\boldsymbol{k}^* \tag{B.4}$$

Using the relation $\boldsymbol{p}^{*T}\boldsymbol{i}^* = \boldsymbol{p}^T\boldsymbol{i}$,

$$\boldsymbol{p}^* = (\boldsymbol{p}^T\boldsymbol{i})\boldsymbol{i}^* + (\boldsymbol{p}^T\boldsymbol{j})\boldsymbol{j}^* + (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k}^* \tag{B.5}$$

Substituting Eq.(B.2) and Eq.(B.3) into Eq.(B.5),

$$\boldsymbol{p}^* = (\boldsymbol{p}^T\boldsymbol{i})(\boldsymbol{i}\cos\alpha + \boldsymbol{j}\sin\alpha) + (\boldsymbol{p}^T\boldsymbol{j})(-\boldsymbol{i}\sin\alpha + \boldsymbol{j}\cos\alpha) + (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k} \tag{B.6}$$

Using the relation $(\boldsymbol{p}^T\boldsymbol{i})\boldsymbol{j} - (\boldsymbol{p}^T\boldsymbol{j})\boldsymbol{i}$=$(\boldsymbol{i} \times \boldsymbol{j}) \times \boldsymbol{p}$=$\boldsymbol{k} \times \boldsymbol{p}$ (see Appendix A),

$$\begin{aligned} \boldsymbol{p}^* &= (\boldsymbol{p}^T\boldsymbol{i})\boldsymbol{i}\cos\alpha + \sin\alpha(\boldsymbol{p}^T\boldsymbol{i})\boldsymbol{j} - \sin\alpha(\boldsymbol{p}^T\boldsymbol{j})\boldsymbol{i} + \cos\alpha(\boldsymbol{p}^T\boldsymbol{j})\boldsymbol{i} + (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k} \\ &= \cos\alpha(\boldsymbol{p} - (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k}) + \sin\alpha(\boldsymbol{k} \times \boldsymbol{p}) + (\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k} \\ &= (1 - \cos\alpha)(\boldsymbol{p}^T\boldsymbol{k})\boldsymbol{k} + \sin\alpha(\boldsymbol{k} \times \boldsymbol{p}) + \cos\alpha\boldsymbol{p} \end{aligned} \tag{B.7}$$

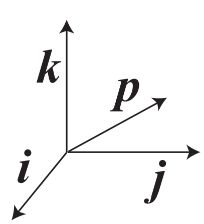


Fig. B.1 Unit vectors i, j, k

Since $\boldsymbol{p}$ is arbitrary, we select $\boldsymbol{p} = \boldsymbol{x} = (1, 0, 0)^T$. Then $\boldsymbol{p}^* = \boldsymbol{x}^*$ is also a unit vector and it is

$$\boldsymbol{x}^* = \cos\alpha \boldsymbol{x} + (\boldsymbol{x}^T\boldsymbol{k})\boldsymbol{k}(1-\cos\alpha) + (\boldsymbol{k}\times\boldsymbol{x})\sin\alpha \tag{B.8}$$

Denoting the $\boldsymbol{k} = [k_x, k_y, k_z]^T$,

$$\boldsymbol{x}^* = \cos\alpha \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + k_x \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} (1-\cos\alpha) + \begin{bmatrix} 0 \\ k_z \\ -k_y \end{bmatrix} \sin\alpha \tag{B.9}$$

Similarly, $\boldsymbol{y}^*$ is

$$\boldsymbol{y}^* = \cos\alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + k_y \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} (1-\cos\alpha) + \begin{bmatrix} -k_z \\ 0 \\ k_x \end{bmatrix} \sin\alpha \tag{B.10}$$

Similary, $\boldsymbol{z}^*$ is

$$\boldsymbol{z}^* = \cos\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + k_z \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} (1-\cos\alpha) + \begin{bmatrix} k_y \\ -k_x \\ 0 \end{bmatrix} \sin\alpha \tag{B.11}$$

By the definition $R = [\boldsymbol{x}^*\ \boldsymbol{y}^*\ \boldsymbol{z}^*]$, we have

$$R = Rot(\boldsymbol{k}, \alpha) =$$

$$\begin{bmatrix} k_x^2(1-\cos\alpha)+\cos\alpha & k_xk_y(1-\cos\alpha)-k_z\sin\alpha & k_zk_x(1-\cos\alpha)+k_y\sin\alpha \\ k_xk_y(1-\cos\alpha)+k_z\sin\alpha & k_y^2(1-\cos\alpha)+\cos\alpha & k_zk_y(1-\cos\alpha)-k_x\sin\alpha \\ k_xk_z(1-\cos\alpha)-k_y\sin\alpha & k_yk_z(1-\cos\alpha)+k_x\cos\alpha & k_z^2(1-\cos\alpha)+\cos\alpha \end{bmatrix} \tag{B.12}$$

Or, by using $v_\alpha = 1-\cos\alpha$, $C_\alpha = \cos\alpha$, $S_\alpha = \sin\alpha$,

$$R =$$

$$\begin{bmatrix} k_x^2v_\alpha + C_\alpha & k_xk_yv_\alpha - k_zS_\alpha & k_zk_xv_\alpha + k_yS_\alpha \\ k_xk_yv_\alpha + k_zS_\alpha & k_y^2v_\alpha + C_\alpha & k_zk_yv_\alpha - k_xS_\alpha \\ k_xk_zv_\alpha - k_yS_\alpha & k_yk_zv_\alpha + k_xC_\alpha & k_z^2v_\alpha + C_\alpha \end{bmatrix} \tag{B.13}$$

Appendix C

Definition of Quaternion and the relation with Rotation Matrix

The quaternion $\boldsymbol{Q}$ has four elements as

$$\boldsymbol{Q} = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} q_0 \\ \boldsymbol{q} \end{bmatrix} = (q_0;\ q_1, q_2, q_3) = (q_0;\ \boldsymbol{q}) = q_0 + q_1\boldsymbol{i} + q_2\boldsymbol{j} + q_3\boldsymbol{k} \tag{C.1}$$

The first element q_0 is called "scalar part" or "real part" and the rest part $\boldsymbol{q}$ is called "vector part" or "imaginary part". The sum and the product for the quaternion is defined as followings.

$$\text{sum} \quad \boldsymbol{Q}+\boldsymbol{P} \quad = (q_0 + p_0;\ \boldsymbol{p}+\boldsymbol{q}) \tag{C.2}$$

$$\text{product} \quad \boldsymbol{QP} \quad = (q_0 p_0 - \boldsymbol{q}\cdot\boldsymbol{p};\ q_0\boldsymbol{p} + p_0\boldsymbol{q} + \boldsymbol{p}\times\boldsymbol{q}) \tag{C.3}$$

Relationship of the quaternion and the rotation around a unit vector $\boldsymbol{k}$ with angle θ is

$$\boldsymbol{Q} = (\cos\frac{\theta}{2};\ \boldsymbol{k}\sin\frac{\theta}{2}) \tag{C.4}$$

Clearly the magnitude of $\boldsymbol{Q}$ is

$$|\boldsymbol{Q}| = \sqrt{\sum_{i=0}^{3} q_i^2} = 1 \tag{C.5}$$

Then the rotation matrix R using the element of $\boldsymbol{Q}$ is described by

$$R(\boldsymbol{Q}) = \begin{bmatrix} q_0^2 - q_1^2 - q_2^2 + q_3^2 & 2(q_0q_1 + q_2q_3) & 2(q_0q_2 - q_1q_3) \\ 2(q_0q_1 - q_2q_3) & -q_0^2 + q_1^2 - q_2^2 + q_3^2 & 2(q_1q_2 - q_0q_3) \\ 2(q_0q_2 + q_1q_3) & 2(q_1q_2 - q_0q_3) & -q_0^2 - q_1^2 + q_2^2 + q_3^2 \end{bmatrix} \tag{C.6}$$

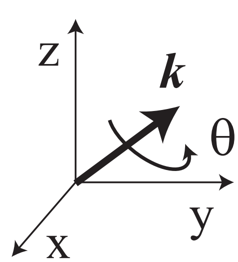


Fig. C.1 Rotation of θ around $\boldsymbol{k}$

On the other hand the element of $\boldsymbol{Q}$ is calculated by the element of R by

$$\begin{aligned} q_3 &= \pm\frac{1}{2}\sqrt{1+R_{11}+R_{12}+R_{33}} && \text{(C.7)} \\ q_0 &= \frac{1}{4q_3}(R_{23}-R_{32}) && \text{(C.8)} \\ q_1 &= \frac{1}{4q_3}(R_{31}-R_{13}) && \text{(C.9)} \\ q_2 &= \frac{1}{4q_3}(R_{12}-R_{21}) && \text{(C.10)} \\ & && \text{(C.11)} \end{aligned}$$

When a vector $\boldsymbol{q}$ is rotated around $\boldsymbol{k}$ with angle θ then $\boldsymbol{q}$ is rotated into $\boldsymbol{p}$ as (see (B.13)),

$$\boldsymbol{p} = R(\boldsymbol{k},\theta)\boldsymbol{q} \tag{C.12}$$

Using the quaternion, we can also calculate

$$\begin{aligned} \boldsymbol{Q} &= (0;\ \boldsymbol{q}), \quad \boldsymbol{P} = (0;\ \boldsymbol{p}) && \text{(C.13)} \\ \boldsymbol{A} &= (\cos\frac{\theta}{2};\ k_x\sin\frac{\theta}{2}, k_y\sin\frac{\theta}{2}, k_z\sin\frac{\theta}{2}) && \text{(C.14)} \\ \boldsymbol{B} &= (\cos\frac{\theta}{2};\ -k_x\sin\frac{\theta}{2}, -k_y\sin\frac{\theta}{2}, -k_z\sin\frac{\theta}{2}) && \text{(C.15)} \\ \boldsymbol{P} &= \boldsymbol{AQB} && \text{(C.16)} \end{aligned}$$

Then $\boldsymbol{p}$ ($P = (0;\boldsymbol{p})$) is the objective vector.

Appendix D

Inertia Tensor and Angular Momentum

When vector $\boldsymbol{p}$ in a rigid body rotates around $\boldsymbol{\omega}$, the velocity of $\boldsymbol{p}$ is represented by

$$\dot{\boldsymbol{p}} = \boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{p} \tag{D.1}$$

By describing the small mass part at point $\boldsymbol{p}$ as dm,

$$\text{momentum for small part} = \boldsymbol{v}dm \tag{D.2}$$
$$\text{angular momentum for small part} = \boldsymbol{p} \times \boldsymbol{v}dm \tag{D.3}$$

For the total rigid body, the angular momentum M is

$$M = \int_V \boldsymbol{p} \times \boldsymbol{v}dm \tag{D.4}$$
$$= \int_V \boldsymbol{p} \times (\boldsymbol{\omega} \times \boldsymbol{p})dm \tag{D.5}$$

By using the formula of vector triple product $\Rightarrow$ matrix times vector,

$$M = \int_V (\boldsymbol{p}^T\boldsymbol{p}E_3 - \boldsymbol{p}\boldsymbol{p}^T)\boldsymbol{\omega}dm$$
$$= \int_V (\boldsymbol{p}^T\boldsymbol{p}E_3 - \boldsymbol{p}\boldsymbol{p}^T)dm\boldsymbol{\omega} \tag{D.6}$$
$$= I\boldsymbol{\omega} \tag{D.7}$$

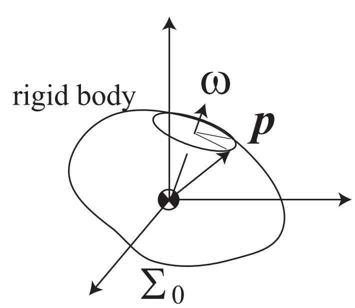


Fig. D.1 Rotating rigid body

where

$$\begin{aligned} I &= \begin{bmatrix} \int_V (p_x^2+p_y^2+p_z^2-p_x^2)dm & -\int_V p_xp_ydm & -\int_V p_xp_zdm \\ -\int_V p_xp_ydm & \int_V (p_x^2+p_y^2+p_z^2-p_y^2)dm & -\int_V p_yp_zdm \\ -\int_V p_xp_zdm & -\int_V p_yp_zdm & \int_V (p_x^2+p_y^2+p_z^2-p_z^2)dm \end{bmatrix} \\ &= \begin{bmatrix} \int_V (p_y^2+p_z^2)dm & -\int_V p_xp_ydm & -\int_V p_xp_zdm \\ -\int_V p_xp_ydm & \int_V (p_x^2+p_z^2)dm & -\int_V p_yp_zdm \\ -\int_V p_xp_zdm & -\int_V p_yp_zdm & \int_V (p_x^2+p_y^2)dm \end{bmatrix} = \begin{bmatrix} I_{xx} & -H_{xy} & -H_{xz} \\ -H_{xy} & I_{yy} & -H_{yz} \\ -H_{xz} & -H_{yz} & I_{zz} \end{bmatrix} \end{aligned} \quad \text{(D.8)}$$

I is called "inertia tensor". The rigid body generally rotates in base coordinate frame Σ_0. This means the element of inertia tensor I changes on time t. This is not favorable. Thus, we next describe the inertia tensor with respect to rigid body coordinate frame to represent the elements of I as constant values. In Σ_A, we have

$$^AM = {}^AI\,^A\boldsymbol{\omega} \quad \text{(D.9)}$$

The momentum M and angular velocity $\boldsymbol{\omega}$ are vectors, thus

$$M = {}^0R_A\,^AM \quad \text{(D.10)}$$

$$\boldsymbol{\omega} = {}^0R_A\,^A\boldsymbol{\omega} \quad \text{(D.11)}$$

Substituting the equations into $M = I\boldsymbol{\omega}$,

$$^0R_A\,^AM = I\,^0R_A\,^A\boldsymbol{\omega} \quad \text{(D.12)}$$

By pre-multiplying $(^0R_A)^{-1} = (^0R_A)^T$ for both side,

$$^AM = (^0R_A)^T\, I\, ^0R_A\,^A\boldsymbol{\omega} \quad \text{(D.13)}$$

Comparing Eq.(D.9) and Eq.(D.13),

$$^AI = (^0R_A)^T\, I\, ^0R_A \quad \text{(D.14)}$$

Or equivalently

$$I = (^0R_A)\,^AI\,(^0R_A)^T \quad \text{(D.15)}$$

This is formula of coordinate transformation for inertia tensor. Note that elements of ^AI are constant even though elements of ^0R_A and I are not constant.

Appendix E

Theorem of Parallel Axes

We next derive the translational transformation for inertia tensor (moment). Consider arbitrary point $\boldsymbol{p}$ in a rigid body. Recall that

$$M = \int_V (\boldsymbol{p}^T \boldsymbol{p} E_3 - \boldsymbol{p}\boldsymbol{p}^T) dm \boldsymbol{\omega} = I\boldsymbol{\omega}$$

Assuming the two coordinate frames Σ_A and Σ_B are parallel and the origin of Σ_A is mass center of rigid body, we now consider the equation in Σ_B by setting $M \to {}^B M$, $\boldsymbol{p} \to {}^B\boldsymbol{p} = ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})$, $\boldsymbol{\omega} \to {}^B\boldsymbol{\omega} = {}^A\boldsymbol{\omega}$, $\int_V {}^A\boldsymbol{p} dm = 0$, then

$$ {}^B M = \int_V ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0}) E_3 - ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T dm {}^B\boldsymbol{\omega} \tag{E.1}$$

Using ${}^B M = {}^B I {}^B\boldsymbol{\omega}$,

$$ {}^B I = \int_V ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0}) E_3 - ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T dm \tag{E.2}$$

The first integral part of right hand side is

$$\begin{aligned} \int_V ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0}) E_3 dm &= \int_V \left\{ {}^A\boldsymbol{p}^T ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0}) - {}^A\boldsymbol{p}_{B0})^T ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0}) \right\} E_3 dm \\ &= \int_V \left\{ ({}^A\boldsymbol{p}^T {}^A\boldsymbol{p}) - 2 {}^A\boldsymbol{p}^T {}^A\boldsymbol{p}_{B0} + ({}^A\boldsymbol{p}_{B0})^T {}^A\boldsymbol{p}_{B0} \right\} E_3 dm \end{aligned} \tag{E.3}$$

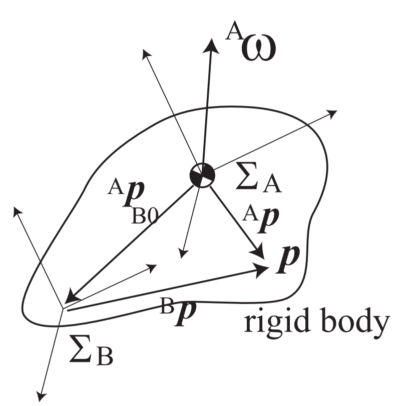


Fig. E.1 Σ_A and Σ_B in a rigid body

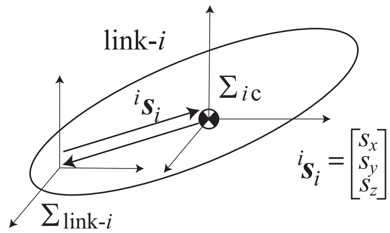


Fig. E.2 Theorem of parallel axes

The second integral part of right hand side is

$$\begin{aligned}\int_V ({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T dm &= \int_V \left\{{}^A\boldsymbol{p}({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T - {}^A\boldsymbol{p}_{B0})({}^A\boldsymbol{p} - {}^A\boldsymbol{p}_{B0})^T\right\} dm \\ &= \int_V \left\{{}^A\boldsymbol{p}{}^A\boldsymbol{p}^T - 2{}^A\boldsymbol{p}{}^A\boldsymbol{p}_{B0}^T + {}^A\boldsymbol{p}_{B0}{}^A\boldsymbol{p}_{B0}^T\right\} dm \qquad \text{(E.4)}\end{aligned}$$

Using the relation $\int_V ({}^A\boldsymbol{p}^T\, {}^A\boldsymbol{p}E_3 - {}^A\boldsymbol{p}{}^A\boldsymbol{p}^T)dm = {}^AI$,

$$\begin{aligned}{}^BI &= {}^AI + \int_V ({}^A\boldsymbol{p}_{B0}^T\, {}^A\boldsymbol{p}_{B0}E_3 - {}^A\boldsymbol{p}_{B0}{}^A\boldsymbol{p}_{B0}^T)dm - 2\int_V ({}^A\boldsymbol{p}^T\, {}^A\boldsymbol{p}_{B0}E_3 - {}^A\boldsymbol{p}{}^A\boldsymbol{p}_{B0}^T)dm \qquad \text{(E.5)} \\ &= {}^AI + {}^A\boldsymbol{p}_{B0}^T\, {}^A\boldsymbol{p}_{B0}E_3 \int_V dm - {}^A\boldsymbol{p}_{B0}{}^A\boldsymbol{p}_{B0}^T \int_V dm - 2\int_V ({}^A\boldsymbol{p}^T\, {}^A\boldsymbol{p}_{B0}E_3 - {}^A\boldsymbol{p}{}^A\boldsymbol{p}_{B0}^T)dm \qquad \text{(E.6)}\end{aligned}$$

where

$$\int_V {}^A\boldsymbol{p}^T\, {}^A\boldsymbol{p}_{B0}E_3 dm = \int_V {}^A\boldsymbol{p}^T dm\, {}^A\boldsymbol{p}_{B0}E_3 = 0 \qquad \text{(E.7)}$$

$$\int_V {}^A\boldsymbol{p}\, {}^A\boldsymbol{p}_{B0}^T dm = \int_V {}^A\boldsymbol{p}dm\, {}^A\boldsymbol{p}_{B0}^T = 0 \qquad \text{(E.8)}$$

thus we have

$${}^BI = {}^AI + ({}^A\boldsymbol{p}_{B0}^T\, {}^A\boldsymbol{p}_{B0}E_3 - {}^A\boldsymbol{p}_{B0}{}^A\boldsymbol{p}_{B0}^T)m \qquad \text{(E.9)}$$

This is called ”theorem of parallel axes”. We also derive another representation using elements.

$$\begin{aligned}{}^iI &= {}^AI + m\left\{\begin{bmatrix} s_x^2+s_y^2+s_z^2 & 0 & \\ 0 & s_x^2+s_y^2+s_z^2 & 0 \\ 0 & 0 & s_x^2+s_y^2+s_z^2 \end{bmatrix} - \begin{bmatrix} s_x^2 & s_xs_y & s_xs_z \\ s_ys_x & s_y^2 & s_ys_z \\ s_zs_x & s_zs_y & s_z^2 \end{bmatrix}\right\} \\ &= {}^AI + m\begin{bmatrix} s_y^2+z_z^2 & -s_xs_y & -s_xs_z \\ -s_ys_x & s_x^2+s_z^2 & -s_ys_z \\ -s_zs_x & -s_zs_y & s_z^2+s_y^2 \end{bmatrix} \qquad \text{(E.10)}\end{aligned}$$

Appendix F

Euler's Equation of Motion

We here prove Euler's equation of motion $N \equiv \frac{d}{dt}(M) = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega}$. Angular momentum M is defined by (see Appendix C)

$$M = I\boldsymbol{\omega} \tag{F.1}$$

We first derive the equation of angular momentum in rigid coordinate frame Σ_A which is attached with center of gravity of rigid body A. Because angular momentum M and moment N are both vectors,

$$^0M = {}^0R_A\, {}^AM \tag{F.2}$$
$$^0N = {}^0R_A\, {}^AN \tag{F.3}$$

Recall the relation

$$^0I = {}^0R_A\, {}^AI({}^0R_A)^T \tag{F.4}$$
$$^0\boldsymbol{\omega} = {}^0R_A\, {}^A\boldsymbol{\omega} \tag{F.5}$$

Using Eq.(F.2)∼Eq.(F.5),

$$^0M = {}^0I_A\, {}^0\boldsymbol{\omega} = {}^0R_A{}^AM \tag{F.6}$$
$$= {}^0R_A\, {}^AI({}^0R_A)^{T0}R_A\, {}^A\boldsymbol{\omega} \tag{F.7}$$
$$= {}^0R_A\, {}^AI^A\boldsymbol{\omega} \tag{F.8}$$

Thus,

$$^AM = {}^AI\, {}^A\boldsymbol{\omega} \tag{F.9}$$

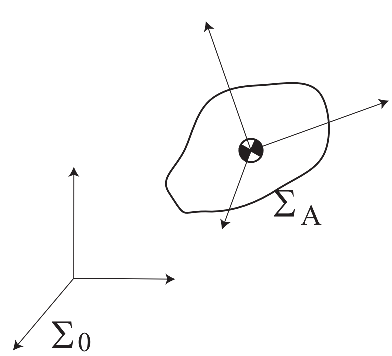


Fig. F.1 Rigid body coordinate frame

This means that the definition of angular momentum $M = I\boldsymbol{\omega}$ also holds in Σ_A. Note that elements of ${}^AI$ are constant values. From Eq.(F.2) and Eq.(F.9),

$$\begin{aligned}\frac{d^0M}{dt} &= {}^0R_A\left(\frac{d^AM}{dt}\right) + {}^0\boldsymbol{\omega}_A \times {}^0R_A\,{}^AM \\ &= {}^0R_A\frac{d}{dt}({}^AI^A\boldsymbol{\omega}) + {}^0\boldsymbol{\omega}_A \times {}^0R_A({}^AI^A\boldsymbol{\omega}) \\ &= {}^0R_A\,{}^AI\frac{d}{dt}({}^A\boldsymbol{\omega}) + {}^0\boldsymbol{\omega}_A \times ({}^0R_A\,{}^AI^A\boldsymbol{\omega}) \qquad \text{(F.10)}\end{aligned}$$

By pre-multiplying ${}^0R_A^T$ with Eq.(F.10),

$$\begin{aligned}({}^0R_A^T)\frac{d^0M}{dt} = ({}^0R_A^T){}^0N &= {}^AI\frac{d}{dt}({}^A\boldsymbol{\omega}) + ({}^0R_A^T)\left[{}^0\boldsymbol{\omega}_A \times ({}^0R_A\,{}^AI^A\boldsymbol{\omega})\right] \\ {}^AN &= {}^AI\frac{d}{dt}({}^A\boldsymbol{\omega}) + {}^AR_0\,{}^0\boldsymbol{\omega}_A \times ({}^0R_A^T\,{}^0R_A\,{}^AI^A\boldsymbol{\omega}) \\ &= {}^AI\frac{d}{dt}({}^A\boldsymbol{\omega}) + {}^A\boldsymbol{\omega} \times ({}^AI^A\boldsymbol{\omega}) \qquad \text{(F.11)}\end{aligned}$$

This equation is Euler's equation of motion in Σ_A. Here we prepare the following relation

$$\begin{aligned}\frac{d}{dt}({}^A\boldsymbol{\omega}) &= \frac{d}{dt}({}^AR_0\,{}^0\boldsymbol{\omega}) \qquad \text{(F.12)} \\ &= {}^A\boldsymbol{\omega} \times {}^AR_0\,{}^0\boldsymbol{\omega} + {}^AR_0\frac{d}{dt}({}^0\boldsymbol{\omega}) \\ &= {}^A\boldsymbol{\omega} \times {}^A\boldsymbol{\omega} + {}^AR_0\frac{d}{dt}({}^0\boldsymbol{\omega}) \qquad \text{(F.13)} \\ &= {}^0R_A^T\frac{d}{dt}({}^0\boldsymbol{\omega}) \qquad \text{(F.14)}\end{aligned}$$

Using the relation ${}^AI = ({}^0R_A)^T\,{}^0I^0R_A$ and Eq.(F.11), Eq.(F.14), we have

$$\begin{aligned}{}^0N = {}^0R_A{}^AN &= {}^0R_A{}^AI\frac{d}{dt}({}^A\boldsymbol{\omega}) + ({}^0R_A){}^A\boldsymbol{\omega} \times ({}^AI^A\boldsymbol{\omega}) \\ &= {}^0R_A(({}^0R_A)^T\,{}^0I^0R_A){}^AI^0R_A^T\frac{d}{dt}({}^0\boldsymbol{\omega}) + ({}^0R_A){}^A\boldsymbol{\omega} \times ({}^AI^A\boldsymbol{\omega}) \\ &= {}^0I\frac{d}{dt}({}^0\boldsymbol{\omega}) + ({}^0R_A){}^A\boldsymbol{\omega} \times (({}^0R_A){}^AI^A\boldsymbol{\omega}) \\ &= {}^0I\frac{d}{dt}({}^0\boldsymbol{\omega}) + ({}^0\boldsymbol{\omega} \times (({}^0R_A)({}^0R_A)^T\,{}^0I^0R_A){}^0R_A^T{}^0\boldsymbol{\omega}) \\ &= {}^0I\frac{d}{dt}({}^0\boldsymbol{\omega}) + {}^0\boldsymbol{\omega} \times {}^0I\,{}^0\boldsymbol{\omega} \qquad \text{(F.15)}\end{aligned}$$

Thus, generally we can describe

$$N = I\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times I\boldsymbol{\omega} \qquad \text{(F.16)}$$

This is called Euler's equation of motion.

Appendix G

Lagrange Equation of Motion

In this appendix, we derive Lagrange equation of motion by analytical mechanics. Consider a mass point $\boldsymbol{x}_j$ in three dimensional space which is a function of generalized coordinate $q_1, \cdots, q_n$ and time t.

$$\boldsymbol{x}_j = \boldsymbol{x}_j(q_1, \cdots, q_n, t) \tag{G.1}$$

where there are h independent constraint conditions for N mass points system. For the system, degrees of freedom n is

$$\text{Degrees of freedom } n = 3N - h \tag{G.2}$$

Then independent n mass points system can be represented by n independent general coordinate $q_1, \cdots .q_n$. For $j-$th mass point,

$$\boldsymbol{F}_j = m_j \ddot{\boldsymbol{x}}_j \tag{G.3}$$

Time derivative of $\boldsymbol{x}_j$ can be written by

$$\begin{aligned} \dot{\boldsymbol{x}}_j &= \frac{\partial \boldsymbol{x}_j}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \boldsymbol{x}_j}{\partial q_n}\dot{q}_n + \frac{\partial \boldsymbol{x}_j}{\partial t} \\ &= \sum_{i=1}^{n} \frac{\partial \boldsymbol{x}_j}{\partial q_i}\dot{q}_i + \frac{\partial \boldsymbol{x}_j}{\partial t} \end{aligned} \tag{G.4}$$

From Eq.(G.4),

$$\frac{\partial \dot{\boldsymbol{x}}_j}{\partial \dot{q}_i} = \frac{\partial \boldsymbol{x}_j}{\partial q_i} \tag{G.5}$$

By taking partial derivative on q_i for Eq.(G.4),

$$\begin{aligned} \frac{\partial \dot{\boldsymbol{x}}_j}{\partial q_i} &= \frac{\partial}{\partial q_i}\left(\frac{\partial \boldsymbol{x}_j}{\partial q_1}\dot{q}_1 + \cdots + \frac{\partial \boldsymbol{x}_j}{\partial q_n}\dot{q}_n + \frac{\partial \boldsymbol{x}_j}{\partial t}\right) \\ &= \frac{\partial}{\partial q_1}\left(\frac{\partial \boldsymbol{x}_j}{\partial q_i}\right)\dot{q}_1 + \cdots + \frac{\partial}{\partial q_n}\left(\frac{\partial \boldsymbol{x}_j}{\partial q_i}\right)\dot{q}_n + \frac{\partial}{\partial t}\frac{\partial \boldsymbol{x}_j}{\partial q_i} \\ &= \frac{d}{dt}\left(\frac{\partial \boldsymbol{x}_j}{\partial q_i}\right) \end{aligned} \tag{G.6}$$

Next, we take partial derivative of $\dot{\boldsymbol{x}}_j^T\, \dot{\boldsymbol{x}}_j$ on $\dot{q}_i$

$$\begin{aligned} \frac{\partial(\dot{\boldsymbol{x}}_j^T\, \dot{\boldsymbol{x}}_j)}{\partial \dot{q}_i} &= \left(\frac{\partial \boldsymbol{x}_j}{\partial \dot{q}_i}\right)^T \dot{\boldsymbol{x}}_j + \dot{\boldsymbol{x}}_j^T\left(\frac{\partial \boldsymbol{x}_j}{\partial \dot{q}_i}\right) \\ &= 2\dot{\boldsymbol{x}}_j^T\,\frac{\partial \boldsymbol{x}_j}{\partial \dot{q}_i} \end{aligned} \tag{G.7}$$

By taking time derivative of Eq.(G.7),

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial(\dot{\boldsymbol{x}}_j^T\,\dot{\boldsymbol{x}}_j)}{\partial\dot{q}_i}\right) &= 2\frac{d}{dt}\left(\dot{\boldsymbol{x}}_j^T\,\frac{\partial\boldsymbol{x}_j}{\partial\dot{q}_i}\right) = 2\left\{\ddot{\boldsymbol{x}}_j^T\frac{\partial\dot{\boldsymbol{x}}_j}{\partial\dot{q}_i} + \dot{\boldsymbol{x}}_j^T\frac{d}{dt}\left(\frac{\partial\dot{\boldsymbol{x}}_j}{\partial\dot{q}_i}\right)\right\} \\ &= 2\left\{\ddot{\boldsymbol{x}}_j^T\frac{\partial\boldsymbol{x}_j}{\partial q_i} + \dot{\boldsymbol{x}}_j^T\frac{\partial\dot{\boldsymbol{x}}_j}{\partial q_i}\right\} \end{aligned} \tag{G.8}$$

where we use Eq.(G.5) and Eq.(G.6). We now take sum of inner product for $\boldsymbol{F}_j = m_j\ddot{\boldsymbol{x}}_j$ and $\dfrac{\partial\boldsymbol{x}_j}{\partial q_i}$,

$$\sum_{j=1}^{n}\boldsymbol{F}_j^T\left(\frac{\partial\boldsymbol{x}_j}{\partial q_i}\right) = \sum_{j=1}^{n}m_j\ddot{\boldsymbol{x}}_j^T\left(\frac{\partial\boldsymbol{x}_j}{\partial q_i}\right) \tag{G.9}$$

By rearranging Eq.(G.8),

$$\ddot{\boldsymbol{x}}_j^T\left(\frac{\partial\boldsymbol{x}_j}{\partial q_i}\right) = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial(\dot{\boldsymbol{x}}_j^T\,\dot{\boldsymbol{x}}_j)}{\partial\dot{q}_i}\right) - \dot{\boldsymbol{x}}_j^T\frac{\partial\dot{\boldsymbol{x}}_j}{\partial q_i}$$

Substituting the equation into Eq.(G.9),

$$\sum_{j=1}^{n}\boldsymbol{F}_j^T\left(\frac{\partial\boldsymbol{x}_j}{\partial q_i}\right) = \sum_{j=1}^{n}m_j\left\{\frac{1}{2}\frac{d}{dt}\left(\frac{\partial(\dot{\boldsymbol{x}}_j^T\,\dot{\boldsymbol{x}}_j)}{\partial\dot{q}_i}\right) - \dot{\boldsymbol{x}}_j^T\frac{\partial\dot{\boldsymbol{x}}_j}{\partial q_i}\right\} \tag{G.10}$$

By denoting left hand of the equation with Q_i and $K = \dfrac{1}{2}\displaystyle\sum_{j=1}^{n}m_j\dot{\boldsymbol{x}}_j^T\,\dot{\boldsymbol{x}}_j$, then we have

$$Q_i = \frac{d}{dt}\left(\frac{\partial K}{\partial\dot{q}_i}\right) - \frac{\partial K}{\partial q_i} \tag{G.11}$$

When we denote conservative force as U_i, U_i does not depend on $\dot{q}_i$, thus $\dfrac{\partial U}{\partial\dot{q}_i} = 0$. By representing Lagrange function $\mathcal{L}$ by $\mathcal{L} = K - U$, then

$$Q_i = \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) - \frac{\partial K}{\partial q_i} \tag{G.12}$$

This is called as Lagrange motion of equation.

Appendix H

Lyapunov Stability Theorem

Generally nonlinear autonomous system (which does not include t explicitly) is described by

$$\dot{\boldsymbol{x}} = \boldsymbol{F}(\boldsymbol{x}) \tag{H.1}$$

We here consider equilibrium point $\boldsymbol{x}_0$ which is satisfied with $\boldsymbol{F}(\boldsymbol{x}_0) = 0$. Then, without loss of generality, we can write

$$\boldsymbol{F}(0) = 0$$

Note that it satisfies by setting $\boldsymbol{x} \leftarrow (\boldsymbol{x} - \boldsymbol{x}_0)$ if $\boldsymbol{x}_0 \neq 0$.

1) Stable : If there exists $^\exists\delta > 0$ satisfying with $||\boldsymbol{x}(0)|| < \delta$ and satisfying $||\boldsymbol{x}(t)|| < \varepsilon$ $(t \geq 0)$ for all trajectories which start from initial point $\boldsymbol{x}(0)$ for $\forall\varepsilon > 0$, then origin 0 is stable.

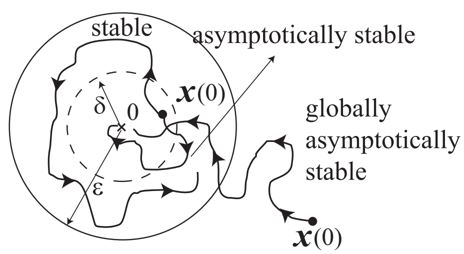


Fig. H.1 Lyapunov stable

2) Asymptotically Stable : If origin 0 is stable and there exists $^\exists\rho < \delta$ satisfying $||\boldsymbol{x}(0)|| < \rho$ and satisfying $\boldsymbol{x}(t) \to 0$ for $t \to \infty$ for trajectory $\boldsymbol{x}(t)$ from any $\boldsymbol{x}(0)$, then origin 0 is asymptotically stable.

3) Globally Asymptotically Stable : If origin 0 is stable and trajectory $\boldsymbol{x}(t)$ from any $\boldsymbol{x}(0)$ is $\boldsymbol{x}(t) \to 0$ for $t \to \infty$, then globally asymptotically stable

We now consider a scalar function $V(\boldsymbol{x})$ such that $V(0) = 0$ and

$$V(\boldsymbol{x}) > 0 \quad (V(\boldsymbol{x}) \text{ is positive definite}) \tag{H.2}$$

For example, quadratic form $V(\boldsymbol{x}) = \boldsymbol{x}^T A \boldsymbol{x}$ is often used. For such case, if $V(\boldsymbol{x})$ is positive definite, then matrix A is a positive definite matrix.

4) Lyapunov function $V(\boldsymbol{x})$:
If $V(\boldsymbol{x})$ is positive definite at $\boldsymbol{x} \in \Omega$, there exists continuous $\dfrac{\partial V}{\partial \boldsymbol{x}}$ and

$$\dot{V}(\boldsymbol{x}) = \frac{dV}{dt} = \frac{\partial V}{\partial \boldsymbol{x}}\frac{d\boldsymbol{x}}{dt} = \frac{\partial V}{\partial \boldsymbol{x}}\boldsymbol{F}(\boldsymbol{x}) \leq 0 \tag{H.3}$$

then $V(\boldsymbol{x})$ is a Lyapunov function.

5) Lyapunov stable theorem:
If there exists a Lyapunov function $V(\boldsymbol{x})$ in the neighborhood Ω of origin 0, then the origin is stable.

5) Lyapunov asymptotically stable theorem:
If Lyapunov stable theorem is satisfied, and

$$\dot{V}(\boldsymbol{x}) < 0 \quad \text{for all } \boldsymbol{x} \neq 0 \tag{H.4}$$

then, the origin is asymptotically stable.

Note that the condition of Lyapunov stable theorem is not necessary and sufficient condition, but a sufficient condition.